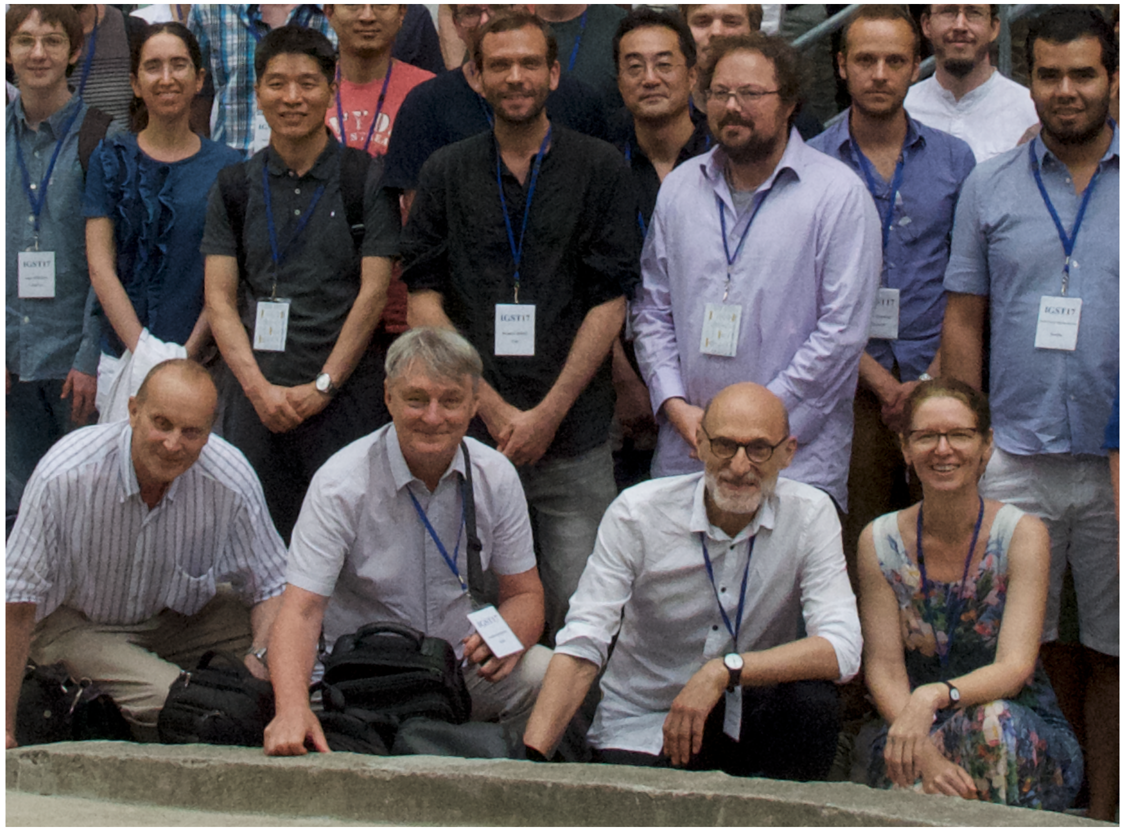
A solvable non-unitary fermionic model with extended symmetry

D. Serban

IPhT Saclay



Integrable Systems and Field Theories
65th Birthday of Fedor Smirnov
Paris, 11-13 October 2023

The model

$$\Delta = \frac{q + q^{-1}}{2}$$

- We are studying at a **XXZ-like deformation** of the Haldane-Shastry spin chain, where the Yangian symmetry can be deformed to a **quantum affine symmetry**

- The model does not possess translational invariance but there is something that replaces it (**quasi-translation invariance**)

- The Hamiltonian is naturally expressed in terms of **Temperley-Lieb** generators

- When the quantum deformation parameter **q is a root of unity** the representations are not isomorphic to the generic ones; for q=i a gl(1|1) structure shows up, we expect gl(2|1) at q^3=1 (the so-called combinatorial point $\Delta = -1/2$)

- q=i is the **free-fermionic point** for XXZ, solvable by Jordan-Wigner. Similar situation here but the boundary conditions render the fermions non-unitary **[Gainutdinov, Read, Saleur, 11]**

- The **even and odd length** chains have **radically different** properties

to appear; w/ A. Ben Moussa, J. Lamers, D.S., A. Toufik

Plan

- Brief reminder of the **isotropic Haldane-Shastry** model
- The **q-deformed Haldane-Shastry** model
- **q=i** limit, **Temperley-Lieb** representation
- Conserved Hamiltonians, **symmetries and spectrum**
- **Free fermions**
- Outlook

The isotropic Haldane-Shastry Hamiltonian

[Haldane, 88; Shastry, 88]

- N **su(2) spins 1/2** on a circle with periodic boundary conditions $z_j \longmapsto \omega^j = \mathrm{e}^{2\pi \mathrm{i} j/N}$

$$H_{\mathrm{HS}} = -\sum_{i \neq j} V(z_i, z_j)\; P_{ij}$$

$$V(z_i, z_j) = \frac{z_i z_j}{(z_i - z_j)^2} = \frac{1}{\sin^2 \pi(i-j)/N}$$

$$P_{jk} = \frac{1}{2}\left(\sigma_j^a \sigma_k^a + 1\right)$$ **spin permutation**

also solvable for su(n) spins in fundamental representation

- **Yangian symmetry** and **2dCFT** limit: [Haldane, Ha, Talstra, Bernard, Pasquier, 92]

algebraic structure: [Bernard, Gaudin, Haldane, Pasquier, 93]

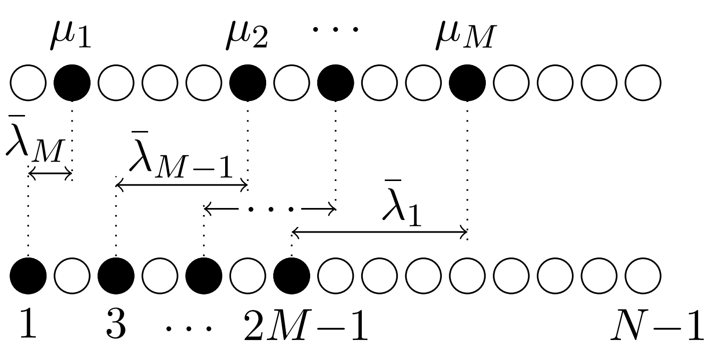
- Yangian and **spinon** description of $su(2)_{k=1}$ CFT: [Bernard, Pasquier, D.S. 94; Bouwknegt, Ludwig, Schoutens, 94]

The spectrum of the Haldane-Shastry Hamiltonian

[Haldane, Ha, Talstra, Bernard, Pasquier, 92]

[Bernard, Gaudin, Haldane, Pasquier, 93]

The model is **Yangian symmetric** (huge degeneracy) and the spectrum is encoded by **motifs**:



M magnon motif

$\bar\lambda_m = \mu_{M-m+1} - 2(M-m) - 1$ $\quad\quad$ $\mu_{m+1} > \mu_m + 1$

"vacuum" M magnon motif

$$E(\mu) - E_0 = \sum_{m=1}^{M} \varepsilon(\mu_m) = \sum_{m=1}^{M} \mu_m (N - \mu_m)$$

each motif corresponds to a **Yangian representation**

"Heisenberg model without bound states"

this structure of the spectrum will be **conserved by the q-deformation** we are considering

The q-Haldane-Shastry Hamiltonian (Uglov-Lamers)

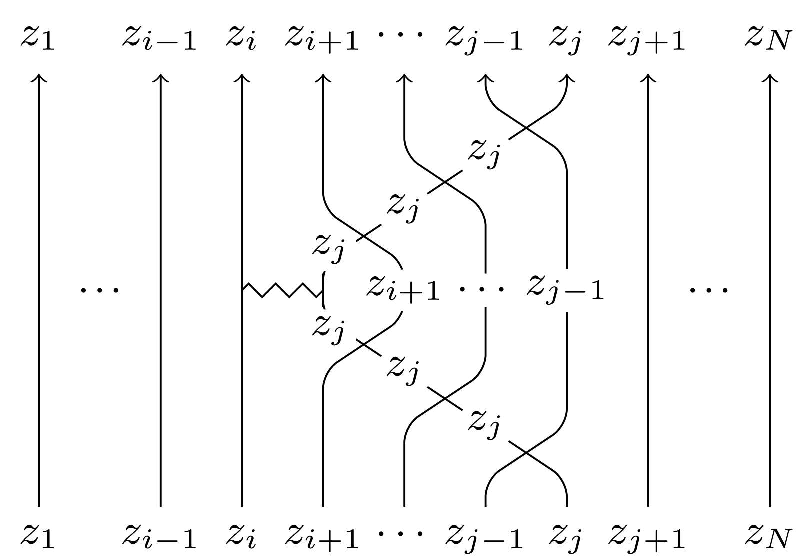
[Bernard, Gaudin, Haldane, Pasquier, 93; Cherednik 92; Uglov 95; Lamers 18; Lamers, Pasquier, D.S., 22]

The XXZ model can also be deformed to accommodate for long-range interaction, at the price of introducing **multi-spin interaction**

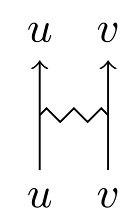
$$\widetilde{\mathrm{H}}^{\mathrm{L}} = \frac{[N]}{N} \sum_{i<j} V(z_i, z_j)\ S^{\mathrm{L}}_{[i,j]} \qquad z_j \longmapsto \omega^j = \mathrm{e}^{2\pi \mathrm{i} j/N}$$

$$V(z_i, z_j) = \frac{z_i\, z_j}{(\mathsf{q}\, z_i - \mathsf{q}^{-1} z_j)(\mathsf{q}^{-1} z_i - \mathsf{q}\, z_j)} \qquad [N] := \frac{\mathsf{q}^N - \mathsf{q}^{-N}}{\mathsf{q} - \mathsf{q}^{-1}}$$

$S^{\mathrm{L}}_{[i,j]} :=$

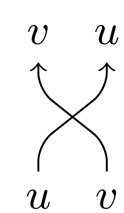


[Lamers 18]



$$:= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathrm{q}^{-1} & -1 & 0 \\ 0 & -1 & \mathrm{q} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = e_i$$

q-antisymmetriser/TL generator



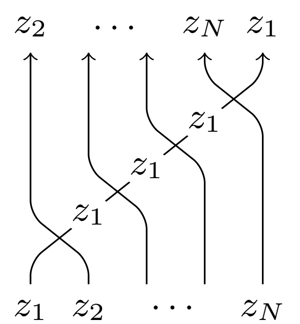
$$:= \check{R}(u/v)$$

$$\check{\mathrm{R}}_{k,k+1}(u) = 1 - f(u)\, e_k\ , \qquad f(u) = \frac{u-1}{\mathrm{q}\, u - \mathrm{q}^{-1}}$$

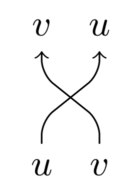
The Uglov-Lamers Hamiltonians

Several new features compared to the case q=1:

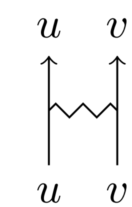
- the model is **not translationally invariant** (but there is a **q-translation operator**, G)

$G =$ 

$$G^N = 1$$

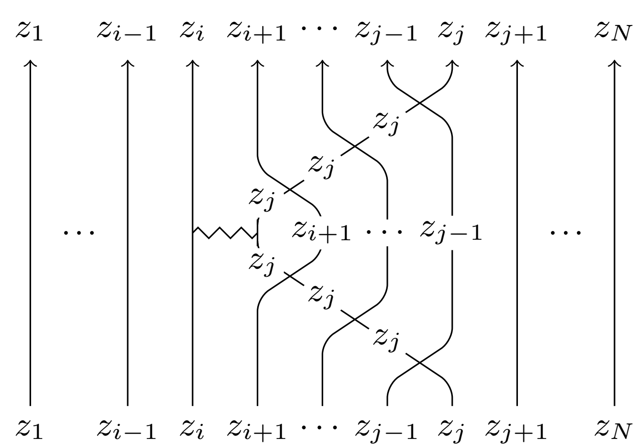


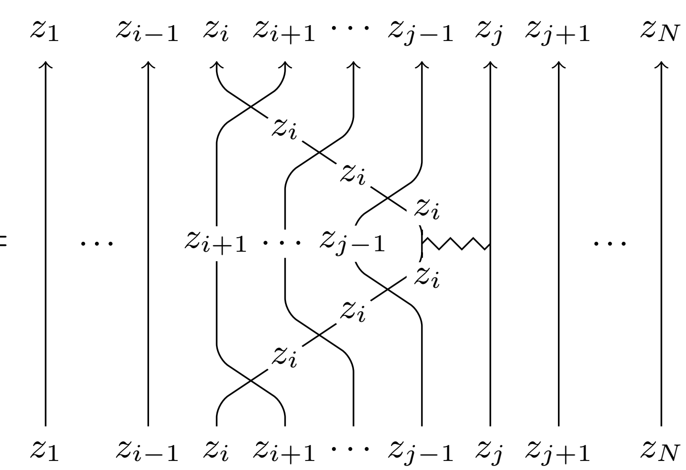
$:= \check{R}(u/v)$



$= e_i$

- there exists **another Hamiltonian with the opposite "chirality"**

$S^{\mathrm{L}}_{[i,j]} :=$ 

$S^{\mathrm{R}}_{[i,j]} :=$  .

$$\widetilde{\mathrm{H}}^{\mathrm{L}} = \frac{[N]}{N} \sum_{i<j} V(z_i, z_j)\, S^{\mathrm{L}}_{[i,j]} \qquad \widetilde{\mathrm{H}}^{\mathrm{R}} = \frac{[N]}{N} \sum_{i<j} V(z_i, z_j)\, S^{\mathrm{R}}_{[i,j]}$$

$$[\widetilde{\mathrm{H}}^{\mathrm{L}}, \widetilde{\mathrm{H}}^{\mathrm{R}}] = 0 \qquad [G, \widetilde{\mathrm{H}}^{\mathrm{L/R}}] = 0$$

The Uglov-Lamers Hamiltonians

$$\widetilde{\mathrm{H}}^{\mathrm{L}} = \frac{[N]}{N} \sum_{i<j} V(z_i, z_j)\ S^{\mathrm{L}}_{[i,j]} \qquad \widetilde{\mathrm{H}}^{\mathrm{R}} = \frac{[N]}{N} \sum_{i<j} V(z_i, z_j)\ S^{\mathrm{R}}_{[i,j]}$$

$$[\widetilde{\mathrm{H}}^{\mathrm{L}}, \widetilde{\mathrm{H}}^{\mathrm{R}}] = 0$$

Both Hamiltonians can be diagonalised simultaneously and the spectrum can be written in terms of **motifs**, with eigenvalues (not real, for |q|=1)

$$\varepsilon^{\mathrm{L,R}}(\mu) = \sum_{m=1}^{M} \varepsilon^{\mathrm{L,R}}(\mu_m)$$

$$\varepsilon^{\mathrm{L}}(n) = \frac{1}{\mathrm{q} - \mathrm{q}^{-1}} \left(\mathrm{q}^{N-n}[n] - \frac{n}{N}[N] \right) , \qquad \varepsilon^{\mathrm{R}}(n) = \frac{-1}{\mathrm{q} - \mathrm{q}^{-1}} \left(\mathrm{q}^{n-N}[n] - \frac{n}{N}[N] \right)$$

$\mathrm{H} = \frac{1}{2}(\widetilde{\mathrm{H}}^{\mathrm{L}} + \widetilde{\mathrm{H}}^{\mathrm{R}})$ has **real spectrum** both for q real and $|\mathrm{q}| = 1$

$$\varepsilon(n) = \frac{1}{2} \left(\varepsilon^{\mathrm{L}}(n) + \varepsilon^{\mathrm{R}}(n) \right) = \frac{1}{2}[n][N-n]$$

q-number generalisation of HS magnon dispersion relation

Temperley-Lieb algebra and the q=i limit

- generators of the **Temperley-Lieb algebra** : $e_j = -h_{[j,j+1]} - \frac{\mathrm{q}-\mathrm{q}^{-1}}{4}(\sigma_j^z - \sigma_{j+1}^z)$

$$e_j^2 = (\mathrm{q}+\mathrm{q}^{-1})\, e_j$$
$$e_j e_{j\pm1} e_j = e_j \,,$$
$$e_j e_k = e_k e_j \qquad (\text{for } j \neq k,\ k\pm 1)\,,$$

XXZ hamiltonian density

- for particular boundary conditions, the open XXZ chain has $U_q sl(2)$ symmetry

[Pasquier, Saleur, 90]

$$H_{\mathrm{XXZ}}^{\mathrm{open}} = \sum_{j=1}^{N-1} h_{[j,j+1]} + \frac{\mathrm{q}-\mathrm{q}^{-1}}{4}(\sigma_1^z - \sigma_N^z) = -\sum_{j=1}^{N-1} e_j$$

- The qHS model is defined in terms of the **XXZ R-matrix**

$$\check{\mathrm{R}}_{k,k+1}(u) = 1 - f(u)\, e_k \,, \qquad f(u) = \frac{u-1}{\mathrm{q}\,u - \mathrm{q}^{-1}}$$

at q=i we have $e_j^2 = 0$ and $f(u^{-1}) = -f(u)$ $\longrightarrow$ great simplification (**free fermions**)

The Uglov-Lamers model at q=i

- in this case the spin interaction can be written exclusively with in terms of **nested commutators of the TL generators**

$$e_{[l,m+1]} := [e_l, [e_{l+1}, \ldots [e_{m-1}, e_m] \ldots]] = [[\ldots [e_l, e_{l+1}], \ldots e_{m-1}], e_m]$$

↑ Jacobi identity and TL algebra

- for example:

$$\mathrm{S}^{\mathrm{L}}_{[i,i+2]} = e_{[i,i+1]} - f^2(\omega) e_{[i+1,i+2]} + f(\omega) e_{[i,i+2]} \,,$$

$$e_{[k,k+1]} \equiv e_k$$

$$\mathrm{S}^{\mathrm{L}}_{[i,i+3]} = e_{[i,i+1]} - f^2(\omega^2) e_{[i+1,i+2]} + f^2(\omega^2) f^2(\omega) e_{[i+2,i+3]} + f(\omega^2) e_{[i,i+2]} - f^2(\omega^2) f(\omega) e_{[i+1,i+3]} + f(\omega^2) f(\omega) e_{[i,i+3]}$$

and in general:

$$\mathrm{S}^{\mathrm{L}}_{[j,j+k]} = \sum_{l=0}^{k-1} \sum_{m=1}^{k-l} (-1)^l \prod_{i=1}^{l} f^2(\omega^{k-i}) \prod_{n=1}^{m-1} f(\omega^{k-l-n}) \; e_{[j+l,j+l+m]}$$

$$\mathrm{S}^{\mathrm{R}}_{[j,j+k]} = \sum_{l=0}^{k-1} \sum_{m=1}^{k-l} (-1)^{l+m-1} \prod_{i=1}^{l} f^2(\omega^{k-i}) \prod_{n=1}^{m-1} f(\omega^{k-l-n}) \; e_{[j+k-l-m,j+k-l]}$$

The Uglov-Lamers model at q=i

There are several subtleties in defining the Hamiltonian at q=i :

since $[2k] = 0$ and $[2k+1] = (-1)^k$

at $N = 2L+1:$ $\varepsilon(n) = \frac{1}{2}[n][N-n] = 0$ the total energy is identically zero

but $\varepsilon^{\mathrm{L}}(n) = -\varepsilon^{\mathrm{R}}(n) = \begin{cases} -n \,, & n = 2k \\ N-n \,, & n = 2k+1 \end{cases}$

the total Hamiltonian is also **zero for odd number of sites**:

$$\mathrm{H} = \sum_{1 \le p \le q < N} \left(h^{\mathrm{L}}_{p,q} + h^{\mathrm{R}}_{p,q}\right) \; e_{[p,q+1]}$$

$$\mathrm{H} = \frac{1}{2}(\widetilde{\mathrm{H}}^{\mathrm{L}} + \widetilde{\mathrm{H}}^{\mathrm{R}})$$

$$h^{\mathrm{L}}_{p,q} = -h^{\mathrm{R}}_{p,q} \,, \qquad 1 \le p \le q < N$$

explicit but tedious expressions/proof

$$h^{\mathrm{L}}_{p,q} = \sum_{k=1}^{N-q} \left(t_{q-p,0}(k) - (-1)^p t_{q-p,p}(k)\right)$$

with

$$t_{p,q}(n) = \prod_{i=0}^{p-1} \tan\frac{\pi(n+i)}{N} \prod_{j=p}^{p+q-1} \tan^2\frac{\pi(n+j)}{N}$$

The Uglov-Lamers model at q=i

One can get a **non vanishing non-chiral Hamiltonian** by expanding to the next order in $\mathrm{q}+\mathrm{q}^{-1}$

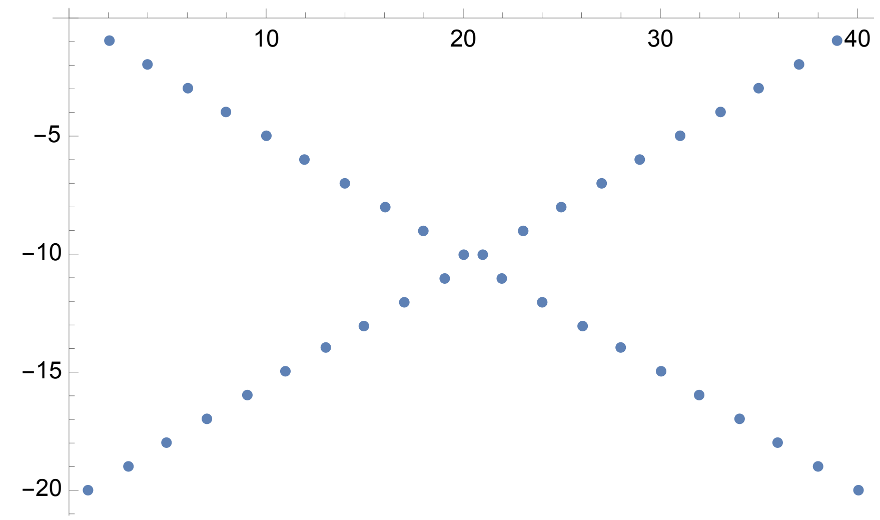
$$\widetilde{\mathrm{H}} := \lim_{\mathrm{q}\to i} \frac{\mathrm{H}}{\mathrm{q}+\mathrm{q}^{-1}}$$

$$[\widetilde{\mathrm{H}}, \widetilde{\mathrm{H}}^{\mathrm{R}}] = -[\widetilde{\mathrm{H}}, \widetilde{\mathrm{H}}^{\mathrm{L}}] = 0$$

- the eigenstates of the chiral and the rescaled Hamiltonians are the same
- the equivalent of the **highest weight vectors** can be constructed using a similar procedure to Haldane-Shastry

result for the **one-magnon dispersion relation**:

$$\tilde{\varepsilon}(n) = \lim_{q\to i} \frac{\varepsilon(n)}{\mathrm{q}+\mathrm{q}^{-1}} = (-1)^{L-1} \begin{cases} \frac{n}{2}\,, & n = 2k \\ \frac{N-n}{2}\,, & n = 2k+1 \end{cases}$$



Th Uglov-Lamers model at q=i

Explicit expression for $\tilde{\mathrm{H}}$?

$N = 5$

99]= $\frac{1}{4}\left(-5-\sqrt{5}\right) e_1 \cdot e_2 + \left(2+\sqrt{5}\right) e_1 \cdot e_3 + \frac{1}{2}\left(-5-\sqrt{5}\right) e_1 \cdot e_4 + \frac{1}{4}\left(-5-\sqrt{5}\right) e_2 \cdot e_1 + \frac{1}{2}\left(2+\sqrt{5}\right) e_2 \cdot e_3 + \left(2+\sqrt{5}\right) e_2 \cdot e_4 +$

$\frac{1}{2}\left(2+\sqrt{5}\right) e_3 \cdot e_2 + \frac{1}{4}\left(-5-\sqrt{5}\right) e_3 \cdot e_4 + \frac{1}{4}\left(-5-\sqrt{5}\right) e_4 \cdot e_3 - \frac{1}{2}\sqrt{5+2\sqrt{5}}\, e_1 \cdot e_2 \cdot e_3 - \sqrt{5+2\sqrt{5}}\, e_1 \cdot e_2 \cdot e_4 +$

$\frac{1}{2}\sqrt{5+2\sqrt{5}}\, e_1 \cdot e_3 \cdot e_2 + \sqrt{5+2\sqrt{5}}\, e_1 \cdot e_3 \cdot e_4 - \sqrt{5+2\sqrt{5}}\, e_1 \cdot e_4 \cdot e_3 - \frac{1}{2}\sqrt{5+2\sqrt{5}}\, e_2 \cdot e_1 \cdot e_3 + \sqrt{5+2\sqrt{5}}\, e_2 \cdot e_1 \cdot e_4 +$

$\frac{1}{2}\sqrt{5+2\sqrt{5}}\, e_2 \cdot e_3 \cdot e_4 + \frac{1}{2}\sqrt{5+2\sqrt{5}}\, e_2 \cdot e_4 \cdot e_3 + \frac{1}{2}\sqrt{5+2\sqrt{5}}\, e_3 \cdot e_2 \cdot e_1 - \frac{1}{2}\sqrt{5+2\sqrt{5}}\, e_3 \cdot e_2 \cdot e_4 - \frac{1}{2}\sqrt{5+2\sqrt{5}}\, e_4 \cdot e_3 \cdot e_2 +$

$\frac{1}{4}\left(3-\sqrt{5}\right) e_1 \cdot e_2 \cdot e_3 \cdot e_4 + \frac{1}{4}\left(-1-\sqrt{5}\right) e_1 \cdot e_2 \cdot e_4 \cdot e_3 + \frac{1}{4}\left(-1+3\sqrt{5}\right) e_1 \cdot e_3 \cdot e_2 \cdot e_4 + \frac{1}{4}\left(-1-\sqrt{5}\right) e_1 \cdot e_4 \cdot e_3 \cdot e_2 +$

$\frac{1}{4}\left(-1-\sqrt{5}\right) e_2 \cdot e_1 \cdot e_3 \cdot e_4 + \frac{1}{4}\left(-1+3\sqrt{5}\right) e_2 \cdot e_1 \cdot e_4 \cdot e_3 + \frac{1}{4}\left(-1-\sqrt{5}\right) e_3 \cdot e_2 \cdot e_1 \cdot e_4 + \frac{1}{4}\left(3-\sqrt{5}\right) e_4 \cdot e_3 \cdot e_2 \cdot e_1$

$$\widetilde{\mathrm{H}} = \sum_{k \le l < m \le n} h_{k,l;m,n} \left\{ e_{[k,l+1]}, e_{[m,n+1]} \right\}$$

explicit coefficients

anti-commutators of nested commutators of TL generators

Non-unitary fermions

- The Temperley-Lieb generators at q=i are expressible in terms of **non-unitary fermions**

$$\{f_j^+, f_k\} = (-1)^j \, \delta_{jk}$$

[Gainutdinov, Read, Saleur, 11]

- Compare with **Jordan-Wigner fermions**: $f_j^+ = (-i)^j \, c_j^+ \,, \qquad f_j = (-i)^j \, c_j$

- Convenient variables: **two-site operators** $g_j = f_j + f_{j+1} \,, \quad g_j^+ = f_j^+ + f_{j+1}^+$

then: $e_j = g_j^+ g_j$

$$e_{[j,j+m+1]} \equiv [[\cdots[e_j, e_{j+1}], \cdots], e_{j+m+1}] = (-1)^{(m-1)j+m(m-1)/2} \left(g_{j+m}^+ \, g_j + (-1)^m g_j^+ \, g_{j+m}\right)$$

quadratic in fermions ⟶ $\widetilde{\mathrm{H}}^{\mathrm{L}}$

$\{e_{[k,l+1]}, e_{[m,n+1]}\}$ **quartic in fermions** ⟶ $\tilde{\mathrm{H}}$

- However $\widetilde{\mathrm{H}}^{\mathrm{L}}$ **not diagonalisable by Fourier transform** (absence of translational invariance)!

⟶ the excitations are fermions dressed with some statistical interaction (**fermionic magnons**)

Fermions and wave functions

$\widetilde{\mathrm{H}}^{\mathrm{L}}$ **quadratic in fermions**

However $\widetilde{\mathrm{H}}^{\mathrm{L}}$ **not diagonalisable by Fourier transform** (absence of translational invariance)

$\longrightarrow$ the excitations are fermions dressed with some statistical interaction (**fermionic magnons**)

$$\Psi_k^+ \equiv G^{1-k}\, f_1^+\, G^{k-1} \propto f_k^+ \,+\, \text{lower}$$

one-magnon state: $$|\{n\}\rangle = \sum_{k=1}^{N} \omega^{nk}\; \Psi_k^+|0\rangle$$

two-magnon highest weight state: $$|\{n_1, n_2\}\rangle = \sum_{k_1<k_2}^{N} P_{n_1,n_2}(\omega^{k_1}, \omega^{k_2})\; \Psi_{k_2}^+ \Psi_{k_1}^+|0\rangle$$

(conjecture)

...

NB: exact expression for the highest weight eigenvectors in the spin language

[Lamers, Pasquier, D.S., 22]

expression of the Hamiltonians in terms of Ψ_k^+?

Even number of sites

The second subtlety appears at ***N* even** $N = 2L$

in this case the dispersion relation is regular (non-vanishing), but there are **divergences** (double poles) in the matrix elements of the Hamiltonians, since:

$$\mathrm{V}_{j,j+L} = \frac{1}{(\mathrm{q}+\mathrm{q}^{-1})^2} \qquad\qquad f(\omega^L) = f(-1) = \frac{2}{\mathrm{q}+\mathrm{q}^{-1}}$$

One of the poles is removed by the factor [N] in the Hamiltonian, but the second has to be removed "by hand" by multiplication with $\mathrm{q}+\mathrm{q}^{-1}$

Result: a **Hamiltonian with finite matrix elements but with identically zero eigenvalues**!

Example: for *N*=2 $\quad 2\mathrm{H} = \frac{1}{\mathrm{q}+\mathrm{q}^{-1}} e_1 \quad$ is a **projector** with eigenvalues 0^3, 1

after rescaling, $\quad 2\mathrm{H}(\mathrm{q}+\mathrm{q}^{-1}) = e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathrm{q}^{-1} & -1 & 0 \\ 0 & -1 & \mathrm{q} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \longrightarrow$ **Jordan block** at q=i

Symmetry

Algebraic origin of the Jordan blocks at N even: **gl(1|1) symmetry**

at each site we have a gl(1|1) representation with alternating central charge E_j

[Gainutdinov, Read, Saleur, 11]

$$\{f_j^+, f_j\} = (-1)^j \equiv E_j \ , \qquad N_j = (-1)^j f_j^+ f_j \ ,$$
$$[\mathrm{N}_j, f_j] = -f_j \ , \qquad [\mathrm{N}_j, f_j^+] = f_j^+ \ .$$

global generators: $F_1^+ = \sum_{j=1}^{N} f_j^+ \ , \qquad F_1 = \sum_{j=1}^{N} f_j \ , \qquad \mathrm{N} = \sum_{j=1}^{N} (-1)^j f_j^+ f_j - L \ , \qquad E = \sum_{j=1}^{N} E_j$

central element

Jordan blocks ⟵ **indecomposable representations** of gl(1|1), at E=0

Experimentally, at larger lengths N=2L, the largest Jordan cell has size L+1

sign of **extended gl(1|1) symmetry**

Conclusions and open questions

- **New fermionic long-range integrable model** with **extended (super)symmetry**
- The **odd and even lengths have very different properties** (linear dispersion relation vs. Jordan blocks)
- Closed form expressions for the (regularised) matrix elements

- Wave functions in the fermionic representation
- Even length chain and interpretation of the Jordan block structure
- **Extended symmetry** of the model in the limit q=i
- Relation with **non-unitary CFTs**
- **Other roots of unity**: q^3=1 and c=0 CFT; gl(2|1) symmetry