Space-like asymptotics of the transverse two-point functions of the XXZ chain at finite temperature

Frank Göhmann and Karol K. Kozlowski

Bergische Universität Wuppertal and ENS de Lyon

Paris, Fedor 65

11.10.2023

Outline of the talk

- Introduction – QStatMech of quantum chains
- Thermal form-factor series (TFFS) for dynamical two-point functions
- Transverse dynamical two-point functions of the XX chain
- Long-time, large-distance asymptotics at fixed finite T directly from the TFFS
- A Fredholm determinant representation and what to learn from it

Outline of the talk

- Introduction – QStatMech of quantum chains
- Thermal form-factor series (TFFS) for dynamical two-point functions
- Transverse dynamical two-point functions of the XX chain
- Long-time, large-distance asymptotics at fixed finite T directly from the TFFS
- A Fredholm determinant representation and what to learn from it
- Karol: Long-time, large-distance asymptotics at fixed finite T directly from the TFFS – XXZ in the critical regime

StatMech of quantum chains

- Quantum (spin) chains

  $\left(\mathbb{C}^d\right)^{\otimes L}$ space of states

  $X_{[\![j,k]\!]} = \mathrm{id}^{\otimes(j-1)} \otimes X \otimes \mathrm{id}^{\otimes(L-k)}, \quad X \in \mathrm{End}\left(\mathbb{C}^d\right)^{\otimes(k-j+1)}$ local operator

  $H_L(c) = \sum_{j=1}^{L} h_{[\![j,j+r-1]\!]}(c) \in \mathrm{End}\left(\mathbb{C}^d\right)^{\otimes L}$ Hamiltonian

StatMech of quantum chains

- Quantum (spin) chains

$(\mathbb{C}^d)^{\otimes L}$	space of states
$X_{[\![j,k]\!]} = \mathrm{id}^{\otimes(j-1)} \otimes X \otimes \mathrm{id}^{\otimes(L-k)}, \quad X \in \mathrm{End}\big(\mathbb{C}^d\big)^{\otimes(k-j+1)}$	local operator
$H_L(c) = \sum_{j=1}^{L} h_{[\![j,j+r-1]\!]}(c) \in \mathrm{End}\big(\mathbb{C}^d\big)^{\otimes L}$	Hamiltonian

- Integrable anthropology

lattice model $\leftrightarrow$ quantum field theory

StatMech of quantum chains

- Quantum (spin) chains

$\left(\mathbb{C}^d\right)^{\otimes L}$ space of states

$X_{[\![j,k]\!]} = \mathrm{id}^{\otimes(j-1)} \otimes X \otimes \mathrm{id}^{\otimes(L-k)}, \quad X \in \mathrm{End}\left(\mathbb{C}^d\right)^{\otimes(k-j+1)}$ local operator

$H_L(c) = \sum_{j=1}^{L} h_{[\![j,j+r-1]\!]}(c) \in \mathrm{End}\left(\mathbb{C}^d\right)^{\otimes L}$ Hamiltonian

- Integrable anthropology

lattice model $\leftrightarrow$ quantum field theory

- QStatMech: response of a large quantum system to time-($= t$)-dependent perturbations (= experiments) described by dynamical correlation functions at finite temperature T

$$\left\langle X_{[\![1,\ell]\!]}(t) Y_{[\![1+m,r+m]\!]} \right\rangle_T = \lim_{L\to\infty} \frac{\mathrm{tr}_{1,\dots,L}\left\{ \mathrm{e}^{(\mathrm{i}t-1/T)H_L}\, X_{[\![1,\ell]\!]}\, \mathrm{e}^{-\mathrm{i}tH_L}\, Y_{[\![1+m,r+m]\!]} \right\}}{\mathrm{tr}_{1,\dots,L}\left\{ \mathrm{e}^{-H_L/T} \right\}}$$

Prime example of an integrable spin chain Hamiltonian

- The XXZ model

$$H_L(\Delta) = J \sum_{j=1}^{L} \left\{ \sigma_{j-1}^x \sigma_j^x + \sigma_{j-1}^y \sigma_j^y + \Delta \sigma_{j-1}^z \sigma_j^z \right\} - \frac{h}{2} \sum_{j=1}^{L} \sigma_j^z$$

$J > 0$, $h \in \mathbb{R}$, $\Delta = \mathrm{ch}(\eta) \in \mathbb{R}$

Prime example of an integrable spin chain Hamiltonian

- The XXZ model

$$H_L(\Delta) = J \sum_{j=1}^{L} \{\sigma_{j-1}^x \sigma_j^x + \sigma_{j-1}^y \sigma_j^y + \Delta \sigma_{j-1}^z \sigma_j^z\} - \frac{h}{2} \sum_{j=1}^{L} \sigma_j^z$$

$J > 0$, $h \in \mathbb{R}$, $\Delta = \mathrm{ch}(\eta) \in \mathbb{R}$

- The big To Do: Calculate

$$\langle \sigma_1^z \sigma_{m+1}^z(t) \rangle_T \,, \quad \langle \sigma_1^- \sigma_{m+1}^+(t) \rangle_T \,, \quad \dots$$

explicitly for all values of m, t, T and h!

Prime example of an integrable spin chain Hamiltonian

- The XXZ model

$$H_L(\Delta) = J \sum_{j=1}^{L} \{ \sigma^x_{j-1}\sigma^x_j + \sigma^y_{j-1}\sigma^y_j + \Delta \sigma^z_{j-1}\sigma^z_j \} - \frac{h}{2} \sum_{j=1}^{L} \sigma^z_j$$

$J > 0$, $h \in \mathbb{R}$, $\Delta = \operatorname{ch}(\eta) \in \mathbb{R}$

- The big To Do: Calculate

$$\langle \sigma^z_1 \sigma^z_{m+1}(t) \rangle_T \,, \quad \langle \sigma^-_1 \sigma^+_{m+1}(t) \rangle_T \,, \quad \dots$$

explicitly for all values of m, t, T and h!

- State of the art: Finite temperature dynamical correlation functions of Yang-Baxter integrable lattice models are largely unknown. Partial exception: the XX model, $H_{XX} = H_L(0)$

Prime example of an integrable spin chain Hamiltonian

- The XXZ model

$$H_L(\Delta) = J \sum_{j=1}^{L} \{ \sigma_{j-1}^x \sigma_j^x + \sigma_{j-1}^y \sigma_j^y + \Delta \sigma_{j-1}^z \sigma_j^z \} - \frac{h}{2} \sum_{j=1}^{L} \sigma_j^z$$

$J > 0$, $h \in \mathbb{R}$, $\Delta = \mathrm{ch}(\eta) \in \mathbb{R}$

- The big To Do: Calculate

$$\langle \sigma_1^z \sigma_{m+1}^z(t) \rangle_T \,, \quad \langle \sigma_1^- \sigma_{m+1}^+(t) \rangle_T \,, \quad \dots$$

explicitly for all values of m, t, T and h!

- State of the art: Finite temperature dynamical correlation functions of Yang-Baxter integrable lattice models are largely unknown. Partial exception: the XX model, $H_{XX} = H_L(0)$

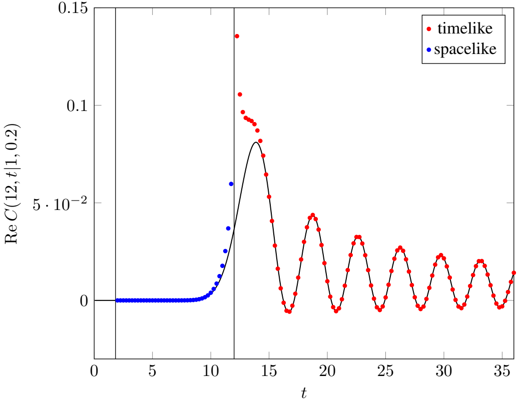
- Longitudinal two-point functions of the XX model [NIEMEIJER 67, GKKKS 17]

$$\langle \sigma_1^z \sigma_{m+1}^z(t) \rangle_T - \langle \sigma_1^z \rangle^2 = \left[\int_{-\pi}^{\pi} \frac{\mathrm{d}p}{\pi} \frac{\mathrm{e}^{\mathrm{i}(mp - t\varepsilon(p))}}{1 + \mathrm{e}^{\varepsilon(p)/T}} \right] \left[\int_{-\pi}^{\pi} \frac{\mathrm{d}p}{\pi} \frac{\mathrm{e}^{-\mathrm{i}(mp - t\varepsilon(p))}}{1 + \mathrm{e}^{-\varepsilon(p)/T}} \right]$$

where $\varepsilon(p) = h - 4J\cos(p)$

Longitudinal correlation functions of the XX model

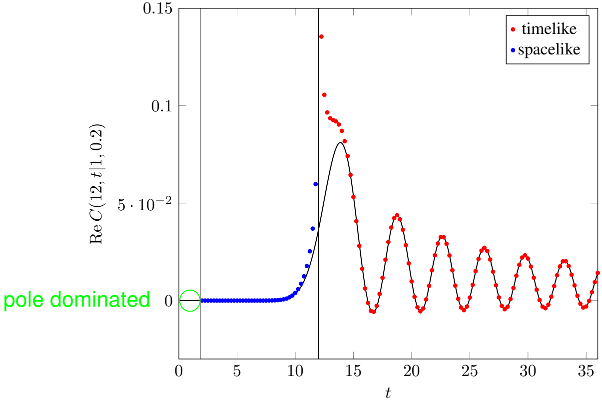
This simple expression can be analyzed numerically and asymptotically by means of the saddle point method



Real part of the connected longitudinal two-point function of the XX chain at $m = 12$, $T = 1$, $h = 0.2$ and $J = 1/4$ as a function of time

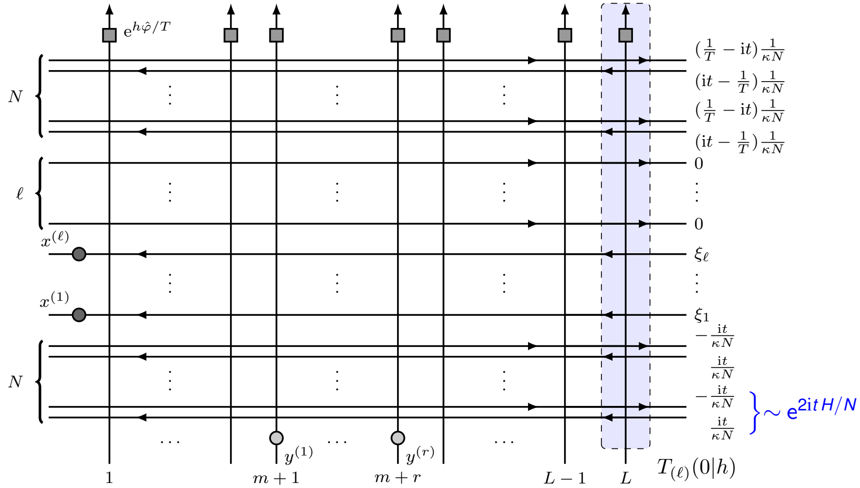
Longitudinal correlation functions of the XX model

This simple expression can be analyzed numerically and asymptotically by means of the saddle point method



Real part of the connected longitudinal two-point function of the XX chain at $m = 12$, $T = 1$, $h = 0.2$ and $J = 1/4$ as a function of time

Lattice path integral representation of dynamical two-point functions



Unnormalized finite Trotter number approximant to the dynamical two-point function, $1/\kappa$ 'energy scale'. $t_{(\ell)}(\lambda|h) = \mathrm{tr}\, T_{(\ell)}(\lambda|h)$ 'dynamical quantum transfer matrix' [SAKAI 07]

A thermal form factor series for dynamical correlation functions

- Expanding (a normalized version of) the above lattice lattice path integral in a basis of eigenstates $|\Psi_k(h)\rangle$ of the dynamical quantum transfer matrix we obtain a 'thermal form factor expansion' of the dynamical two-point functions [GKKKS 17, GKM 23]

A thermal form factor series for dynamical correlation functions

- Expanding (a normalized version of) the above lattice lattice path integral in a basis of eigenstates $|\Psi_k(h)\rangle$ of the dynamical quantum transfer matrix we obtain a 'thermal form factor expansion' of the dynamical two-point functions [GKKKS 17, GKM 23]
- It contains two kinds of objects

A thermal form factor series for dynamical correlation functions

- Expanding (a normalized version of) the above lattice lattice path integral in a basis of eigenstates $|\Psi_k(h)\rangle$ of the dynamical quantum transfer matrix we obtain a 'thermal form factor expansion' of the dynamical two-point functions [GKKKS 17, GKM 23]
- It contains two kinds of objects
- The thermal form factors

$$F_k^{(-)}(X) = \frac{\langle \Psi_0(h)| \prod_{k \in [\![1,\ell]\!]}^{\curvearrowright} \mathrm{tr}_0 \{ x_0^{(k)} T(0|h) \} |\Psi_k(h)\rangle}{\langle \Psi_0(h)|\Psi_0(h)\rangle \Lambda_k^{\ell}(0|h)}$$

$$F_k^{(+)}(Y) = \frac{\langle \Psi_k(h)| \prod_{k \in [\![1,r]\!]}^{\curvearrowright} \mathrm{tr}_0 \{ y_0^{(k)} T(0|h) \} |\Psi_0(h)\rangle}{\langle \Psi_k(h)|\Psi_k(h)\rangle \Lambda_0^{r}(0|h)}$$

that can be characterized by discrete rqKZ [AK 11], by their connection to the Fermionic basis [BJMST 08, JMS 09, BJMS 10], or multiple integrals [BG 09]

A thermal form factor series for dynamical correlation functions

- Expanding (a normalized version of) the above lattice lattice path integral in a basis of eigenstates $|\Psi_k(h)\rangle$ of the dynamical quantum transfer matrix we obtain a 'thermal form factor expansion' of the dynamical two-point functions [GKKKS 17, GKM 23]
- It contains two kinds of objects
- The thermal form factors

$$F_k^{(-)}(X) = \frac{\langle \Psi_0(h)| \prod^{\curvearrowright}_{k \in [\![1,\ell]\!]} \mathrm{tr}_0\{x_0^{(k)} T(0|h)\} |\Psi_k(h)\rangle}{\langle \Psi_0(h)|\Psi_0(h)\rangle \Lambda_k^\ell(0|h)}$$

$$F_k^{(+)}(Y) = \frac{\langle \Psi_k(h)| \prod^{\curvearrowright}_{k \in [\![1,r]\!]} \mathrm{tr}_0\{y_0^{(k)} T(0|h)\} |\Psi_0(h)\rangle}{\langle \Psi_k(h)|\Psi_k(h)\rangle \Lambda_0^r(0|h)}$$

that can be characterized by discrete rqKZ [AK 11], by their connection to the Fermionic basis [BJMST 08, JMS 09, BJMS 10], or multiple integrals [BG 09]

- Ratios of eigenvalues $\Lambda_k(\lambda|h)$ of the dynamical QTM

$$\rho_k(\lambda) = \frac{\Lambda_k(\lambda|h)}{\Lambda_0(\lambda|h)}$$

A thermal form factor series for dynamical correlation functions

- With the above notation the thermal form factor series is

$$\big\langle X_{[\![1,\ell]\!]}(t)\, Y_{[\![1+m,r+m]\!]}\big\rangle_T = \mathrm{e}^{-\mathrm{i}hts(X)} \lim_{N\to\infty} \sum_k A_k^{XY}\, \rho_k(0)^m \left(\frac{\rho_k\big(\frac{-\mathrm{i}t}{\kappa N}\big)}{\rho_k\big(\frac{\mathrm{i}t}{\kappa N}\big)} \right)^{\frac{N}{2}} \qquad \text{(TFFS)}$$

where $A_k^{XY} = F_k^{(-)}(X) F_k^{(+)}(Y)$ and $s(X)$ is the $U(1)$ charge of X

A thermal form factor series for dynamical correlation functions

- With the above notation the thermal form factor series is

$$\big\langle X_{[\![1,\ell]\!]}(t) Y_{[\![1+m,r+m]\!]} \big\rangle_T = \mathrm{e}^{-\mathrm{i}hts(X)} \lim_{N\to\infty} \sum_k A_k^{XY} \rho_k(0)^m \left(\frac{\rho_k(\frac{-\mathrm{i}t}{\kappa N})}{\rho_k(\frac{\mathrm{i}t}{\kappa N})} \right)^{\frac{N}{2}} \qquad \text{(TFFS)}$$

  where $A_k^{XY} = F_k^{(-)}(X) F_k^{(+)}(Y)$ and $s(X)$ is the $U(1)$ charge of X

- The eigenvalues of the quantum transfer matrix are generically non-real and, in the Trotter limit $N \to \infty$ stabilize to a sequence that converges to zero. Eigenvalues ratios $\rho_k(0)$ are real or come in complex conjugate pairs. They can be ordered such that $|\rho_k(0)|$ is a monotonically decreasing sequence that converges to zero

A thermal form factor series for dynamical correlation functions

- With the above notation the thermal form factor series is

$$\langle X_{[\![1,\ell]\!]}(t) Y_{[\![1+m,r+m]\!]} \rangle_T = \mathrm{e}^{-\mathrm{i}hts(X)} \lim_{N\to\infty} \sum_k A_k^{XY} \rho_k(0)^m \left(\frac{\rho_k\left(\frac{-\mathrm{i}t}{\kappa N}\right)}{\rho_k\left(\frac{\mathrm{i}t}{\kappa N}\right)} \right)^{\frac{N}{2}} \qquad \text{(TFFS)}$$

  where $A_k^{XY} = F_k^{(-)}(X) F_k^{(+)}(Y)$ and $s(X)$ is the $U(1)$ charge of X

- The eigenvalues of the quantum transfer matrix are generically non-real and, in the Trotter limit $N \to \infty$ stabilize to a sequence that converges to zero. Eigenvalues ratios $\rho_k(0)$ are real or come in complex conjugate pairs. They can be ordered such that $|\rho_k(0)|$ is a monotonically decreasing sequence that converges to zero

- For large N the first term in the series takes the form

$$A_1^{XY} \exp\left\{ -\mathrm{i}hts(X) + \ln(\rho_1(0))m - \frac{\rho_1'(0)}{\rho_1(0)} \cdot \frac{\mathrm{i}t}{\kappa} \right\} \qquad (*)$$

A thermal form factor series for dynamical correlation functions

- With the above notation the thermal form factor series is

$$\langle X_{[\![1,\ell]\!]}(t) Y_{[\![1+m,r+m]\!]} \rangle_T = \mathrm{e}^{-\mathrm{i}hts(X)} \lim_{N\to\infty} \sum_k A_k^{XY} \rho_k(0)^m \left(\frac{\rho_k(\frac{-\mathrm{i}t}{\kappa N})}{\rho_k(\frac{\mathrm{i}t}{\kappa N})} \right)^{\frac{N}{2}} \qquad \text{(TFFS)}$$

  where $A_k^{XY} = F_k^{(-)}(X) F_k^{(+)}(Y)$ and $s(X)$ is the $U(1)$ charge of X

- The eigenvalues of the quantum transfer matrix are generically non-real and, in the Trotter limit $N \to \infty$ stabilize to a sequence that converges to zero. Eigenvalues ratios $\rho_k(0)$ are real or come in complex conjugate pairs. They can be ordered such that $|\rho_k(0)|$ is a monotonically decreasing sequence that converges to zero

- For large N the first term in the series takes the form

$$A_1^{XY} \exp\left\{ -\mathrm{i}hts(X) + \ln(\rho_1(0))m - \frac{\rho_1'(0)}{\rho_1(0)} \cdot \frac{\mathrm{i}t}{\kappa} \right\} \qquad (*)$$

- In general, all terms in (TFFS) contribute to the large m, t asymptotics, and we have to take the limit $N \to \infty$ first

A thermal form factor series for dynamical correlation functions

- With the above notation the thermal form factor series is

$$\big\langle X_{[\![1,\ell]\!]}(t)\, Y_{[\![1+m,r+m]\!]}\big\rangle_T = \mathrm{e}^{-\mathrm{i}hts(X)} \lim_{N\to\infty} \sum_k A_k^{XY} \rho_k(0)^m \left(\frac{\rho_k\big(\frac{-\mathrm{i}t}{\kappa N}\big)}{\rho_k\big(\frac{\mathrm{i}t}{\kappa N}\big)}\right)^{\frac{N}{2}} \qquad \text{(TFFS)}$$

  where $A_k^{XY} = F_k^{(-)}(X) F_k^{(+)}(Y)$ and $s(X)$ is the $U(1)$ charge of X

- The eigenvalues of the quantum transfer matrix are generically non-real and, in the Trotter limit $N \to \infty$ stabilize to a sequence that converges to zero. Eigenvalues ratios $\rho_k(0)$ are real or come in complex conjugate pairs. They can be ordered such that $|\rho_k(0)|$ is a monotonically decreasing sequence that converges to zero

- For large N the first term in the series takes the form

$$A_1^{XY} \exp\left\{-\mathrm{i}hts(X) + \ln(\rho_1(0))m - \frac{\rho_1'(0)}{\rho_1(0)} \cdot \frac{\mathrm{i}\alpha m}{\kappa}\right\} \qquad (*)$$

- In general, all terms in (TFFS) contribute to the large m, t asymptotics, and we have to take the limit $N \to \infty$ first

- Still, we may ask: When does $(*)$ give the the leading term for large m, t?

Trotter limit $N \to \infty$

- The above is valid for all fundamental Yang-Baxter integrable models. For the evaluation of the series in the Trotter limit we need
  1. access to the full spectrum of the dynamical quantum transfer matrix
  2. to be able to efficiently calculate matrix elements

  This restricts us for the moment to models related to the six-vertex model

Trotter limit $N \to \infty$

- The above is valid for all fundamental Yang-Baxter integrable models. For the evaluation of the series in the Trotter limit we need
  1. access to the full spectrum of the dynamical quantum transfer matrix
  2. to be able to efficiently calculate matrix elements

  This restricts us for the moment to models related to the six-vertex model

- Spectral analysis

$$\mathfrak{a}_N(\lambda|\{\lambda^p,\lambda^h\}) = \frac{d_N(\lambda)}{a_N(\lambda)} \frac{Q(\lambda+\eta|\{\lambda^p,\lambda^h\})}{Q(\lambda-\eta|\{\lambda^p,\lambda^h\})}$$

$$\ln \mathfrak{a}_N(\lambda|\{\lambda^p,\lambda^h\}) = -\frac{\mathfrak{w}_N(\lambda)}{T} + \Phi(\lambda|\{\lambda^p,\lambda^h\}) - [\hat{K}_{\mathcal{C}} \ln(1+\mathfrak{a}_N(\cdot|\{\lambda^p,\lambda^h\}))](\lambda)$$

$$\mathfrak{a}_N(\lambda_j^{p/h}|\{\lambda^p,\lambda^h\}) = -1$$

Trotter limit $N \to \infty$

- The above is valid for all fundamental Yang-Baxter integrable models. For the evaluation of the series in the Trotter limit we need
  1. access to the full spectrum of the dynamical quantum transfer matrix
  2. to be able to efficiently calculate matrix elements

  This restricts us for the moment to models related to the six-vertex model

- Spectral analysis

$$\mathfrak{a}_N(\lambda|\{\lambda^p,\lambda^h\}) = \frac{d_N(\lambda)}{a_N(\lambda)} \frac{Q(\lambda+\eta|\{\lambda^p,\lambda^h\})}{Q(\lambda-\eta|\{\lambda^p,\lambda^h\})}$$

$$\ln \mathfrak{a}_N(\lambda|\{\lambda^p,\lambda^h\}) = -\frac{\mathfrak{w}_N(\lambda)}{T} + \Phi(\lambda|\{\lambda^p,\lambda^h\}) - [\hat{K}_{\mathcal{C}} \ln(1+\mathfrak{a}_N(\cdot|\{\lambda^p,\lambda^h\}))](\lambda)$$

$$\mathfrak{a}_N(\lambda_j^{p/h}|\{\lambda^p,\lambda^h\}) = -1$$

- $N \to \infty$: $\mathfrak{w}_N \to \varepsilon_0(\lambda)$ independent of t $\Rightarrow$ $\mathfrak{a}_N \to \mathfrak{a}$ independent of t

Trotter limit $N \to \infty$

- The above is valid for all fundamental Yang-Baxter integrable models. For the evaluation of the series in the Trotter limit we need
  1. access to the full spectrum of the dynamical quantum transfer matrix
  2. to be able to efficiently calculate matrix elements

  This restricts us for the moment to models related to the six-vertex model

- Spectral analysis

$$\mathfrak{a}_N(\lambda|\{\lambda^p,\lambda^h\}) = \frac{d_N(\lambda)}{a_N(\lambda)} \frac{Q(\lambda+\eta|\{\lambda^p,\lambda^h\})}{Q(\lambda-\eta|\{\lambda^p,\lambda^h\})}$$

$$\ln \mathfrak{a}_N(\lambda|\{\lambda^p,\lambda^h\}) = -\frac{\mathfrak{w}_N(\lambda)}{T} + \Phi(\lambda|\{\lambda^p,\lambda^h\}) - [\hat{K}_{\mathcal{C}} \ln(1+\mathfrak{a}_N(\cdot|\{\lambda^p,\lambda^h\}))](\lambda)$$

$$\mathfrak{a}_N(\lambda_j^{p/h}|\{\lambda^p,\lambda^h\}) = -1$$

- $N \to \infty$: $\mathfrak{w}_N \to \varepsilon_0(\lambda)$ independent of t $\Rightarrow$ $\mathfrak{a}_N \to \mathfrak{a}$ independent of t

- Summation of the TFFS: Put $\lambda^{p/h}$ in off-shell position, use multiple-residue calculus with off-shell version of $\mathfrak{a}$ for summation

Functions characterizing XX, $\Delta = 0$

- $\Delta = 0 \Rightarrow \hat{K}_{\mathcal{C}} = 0, \Phi = 0$ and in the Trotter limit

$$\mathfrak{a}_N(\lambda|\{\lambda^p,\lambda^h\}) = (-1)^s \, \mathrm{e}^{-\frac{h}{T}} \prod_{j=1}^{N} \mathrm{e}^{\mathrm{i}\left(p(\lambda-\nu_{2j-1})-p(\lambda-\nu_{2j})\right)} \to (-1)^s \, \mathrm{e}^{-\frac{\varepsilon(\lambda)}{T}}$$

where $s = n_h - n_p$, and p and ε are one-particle momentum and energy

$$p(\lambda) = -\mathrm{i}\ln\bigl(-\mathrm{i}\,\mathrm{th}(\lambda)\bigr)\,, \quad \varepsilon(\lambda) = h + 2Jp'(\lambda)$$

Functions characterizing XX, $\Delta = 0$

- $\Delta = 0 \ \Rightarrow\ \hat{K}_{\mathcal{C}} = 0, \Phi = 0$ and in the Trotter limit

$$\mathfrak{a}_N(\lambda|\{\lambda^p,\lambda^h\}) = (-1)^s \, e^{-\frac{h}{T}} \prod_{j=1}^{N} e^{i\left(p(\lambda-\nu_{2j-1})-p(\lambda-\nu_{2j})\right)} \ \to\ (-1)^s \, e^{-\frac{\varepsilon(\lambda)}{T}}$$

  where $s = n_h - n_p$, and p and ε are one-particle momentum and energy

$$p(\lambda) = -i \ln\left(-i \, \mathrm{th}(\lambda)\right), \quad \varepsilon(\lambda) = h + 2Jp'(\lambda)$$

- These functions live in the fundamental strip $\mathcal{S} = \left\{\lambda \in \mathbb{C} \middle| -\frac{\pi}{4} \le \mathrm{Im}\,\lambda < \frac{3\pi}{4}\right\}$, where the one-particle energy ε has precisely two roots $\lambda_F^{\pm}$, the Fermi rapidities. $p_F = p(\lambda_F^-) = \arccos(h/4J)$ is the called the Fermi momentum

Functions characterizing XX, $\Delta = 0$

- $\Delta = 0 \Rightarrow \hat{K}_{\mathcal{C}} = 0, \Phi = 0$ and in the Trotter limit

$$\mathfrak{a}_N(\lambda|\{\lambda^p,\lambda^h\}) = (-1)^s \mathrm{e}^{-\frac{h}{T}} \prod_{j=1}^{N} \mathrm{e}^{\mathrm{i}\left(p(\lambda-\nu_{2j-1})-p(\lambda-\nu_{2j})\right)} \rightarrow (-1)^s \mathrm{e}^{-\frac{\varepsilon(\lambda)}{T}}$$

where $s = n_h - n_p$, and p and ε are one-particle momentum and energy

$$p(\lambda) = -\mathrm{i}\ln\left(-\mathrm{i}\,\mathrm{th}(\lambda)\right), \quad \varepsilon(\lambda) = h + 2Jp'(\lambda)$$

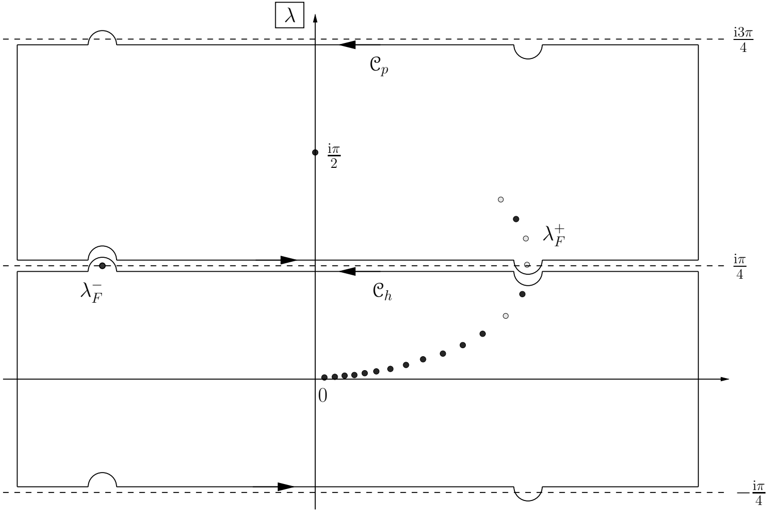
- These functions live in the fundamental strip $\mathcal{S} = \left\{\lambda \in \mathbb{C} \middle| -\frac{\pi}{4} \leq \mathrm{Im}\,\lambda < \frac{3\pi}{4}\right\}$, where the one-particle energy ε has precisely two roots $\lambda_F^{\pm}$, the Fermi rapidities. $p_F = p(\lambda_F^-) = \arccos(h/4J)$ is the called the Fermi momentum

- The eigenvalue ratios take the form

$$\rho(\lambda|\{\lambda^p,\lambda^h\}) =$$

$$(-1)^s \exp\left\{\mathrm{i}\sum_{k=1}^{n_h} p(\lambda_k^h-\lambda) - \mathrm{i}\sum_{k=1}^{n_p} p(\lambda_k^p-\lambda) + \int_{\mathcal{C}_h} \frac{\mathrm{d}\mu}{2\pi} p'(\mu-\lambda) \ln\left(\frac{1+(-1)^s \mathrm{e}^{-\frac{\varepsilon(\mu)}{T}}}{1+\mathrm{e}^{-\frac{\varepsilon(\mu)}{T}}}\right)\right\}$$

Contours for XX



Particle and hole contours $\mathcal{C}_p$ and $\mathcal{C}_h$

Functions occurring in the TFFS for XX

- An auxiliary function

$$z(\lambda) = \frac{1}{2\pi \mathrm{i}} \ln\left\{ \mathrm{cth}\left(\frac{\varepsilon(\lambda)}{2T} \right) \right\}$$

- For all $x \in \mathcal{S} \setminus \mathcal{C}_{h/p}$ two functions

$$\Phi_{h/p}(x) = \frac{\mathrm{i}p'(x)}{2} \exp\left\{ \pm \mathrm{i} \int_{\mathcal{C}_{h/p}} \mathrm{d}\lambda\, p'(\lambda) z(\lambda) \frac{\mathrm{sh}(x+\lambda)}{\mathrm{sh}(x-\lambda)} \right\}$$

- An amplitude

$$\mathcal{A}(T,h) = \exp\left\{ - \int_{\mathcal{C}'_h \subset \mathcal{C}_h} \mathrm{d}\lambda\, z(\lambda) \int_{\mathcal{C}_h} \mathrm{d}\mu\, z(\mu)\, \mathrm{cth}'(\lambda - \mu) \right\}$$

- Square of a generalized Cauchy determinant

$$\mathcal{D}\left(\{x_j\}_{j=1}^{n_h}, \{y_k\}_{k=1}^{n_p} \right) = \frac{\left[\prod_{1 \le j < k \le n_h} \mathrm{sh}^2(x_j - x_k) \right] \left[\prod_{1 \le j < k \le n_p} \mathrm{sh}^2(y_j - y_k) \right]}{\prod_{j=1}^{n_h} \prod_{k=1}^{n_p} \mathrm{sh}^2(x_j - y_k)}$$

TFFS for the transverse dynamical two-point function of the XX chain

- THEOREM: [GKKKS 17] The transverse dynamical two-point function of the XX chain has the series representation

$$\langle \sigma_1^- \sigma_{m+1}^+(t) \rangle_T = (-1)^m \mathcal{F}(m) \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n-1)!} \prod_{j=1}^{n} \int_{\mathcal{C}_h} \frac{\mathrm{d}x_j}{\pi \mathrm{i}} \frac{\Phi_p(x_j)\, \mathrm{e}^{\mathrm{i}(mp(x_j) - t\varepsilon(x_j))}}{1 - \mathrm{e}^{\varepsilon(x_j)/T}}$$

$$\times \prod_{k=1}^{n-1} \int_{\mathcal{C}_p} \frac{\mathrm{d}y_k}{\pi \mathrm{i}} \frac{\mathrm{e}^{-\mathrm{i}(mp(y_k) - t\varepsilon(y_k))}}{\Phi_h(y_k)\big(1 - \mathrm{e}^{-\varepsilon(y_k)/T}\big)} \mathcal{D}\big(\{x_j\}_{j=1}^n, \{y_k\}_{k=1}^{n-1}\big)$$

where

$$\mathcal{F}(m) = \mathrm{e}^{-\mathrm{i}mp_F} \mathcal{A}(T,h) \exp\left\{ -m \int_{\mathcal{C}_h} \frac{\mathrm{d}\lambda}{2\pi} p'(\lambda) \ln \left| \mathrm{cth}\left(\frac{\varepsilon(\lambda)}{2T} \right) \right| \right\}$$

TFFS for the transverse dynamical two-point function of the XX chain

- THEOREM: [GKKKS 17] The transverse dynamical two-point function of the XX chain has the series representation

$$\langle \sigma_1^- \sigma_{m+1}^+(t) \rangle_T = (-1)^m \mathcal{F}(m) \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n-1)!} \prod_{j=1}^{n} \int_{\mathcal{C}_h} \frac{\mathrm{d}x_j}{\pi \mathrm{i}} \frac{\Phi_p(x_j)\, \mathrm{e}^{\mathrm{i}(mp(x_j) - t\varepsilon(x_j))}}{1 - \mathrm{e}^{\varepsilon(x_j)/T}}$$
$$\times \prod_{k=1}^{n-1} \int_{\mathcal{C}_p} \frac{\mathrm{d}y_k}{\pi \mathrm{i}} \frac{\mathrm{e}^{-\mathrm{i}(mp(y_k) - t\varepsilon(y_k))}}{\Phi_h(y_k)\big(1 - \mathrm{e}^{-\varepsilon(y_k)/T}\big)} \mathcal{D}\big(\{x_j\}_{j=1}^n, \{y_k\}_{k=1}^{n-1}\big)$$

  where

$$\mathcal{F}(m) = \mathrm{e}^{-\mathrm{i}mp_F} \mathcal{A}(T,h) \exp\left\{ -m \int_{\mathcal{C}_h} \frac{\mathrm{d}\lambda}{2\pi} p'(\lambda) \ln \left| \mathrm{cth}\left(\frac{\varepsilon(\lambda)}{2T} \right) \right| \right\}$$

- This is now a series over classes of excitations, just a function to be further studied
- We claim that the series is manifestly different from the series obtained by [IIKS 93] and rather appropriate for numerical and asymptotic analysis

Spacelike asymptotics for fixed T

- THEOREM: [GKS 19] In the spacelike regime $m > 4Jt$ our TFFS for the transversal two-point function is absolutely convergent and determines the late-time long-distance asymptotics

$$t \to +\infty, \quad m \to +\infty \quad \text{at any fixed ratio } \alpha = \frac{m}{4Jt} > 1$$

of the transverse dynamical correlation function of the XX chain explicitly

$$\langle \sigma_1^- \sigma_{m+1}^+(t) \rangle_T = C(T,h)(-1)^m \exp\left\{ -m \int_{\mathcal{C}_h} \frac{d\lambda}{2\pi} p'(\lambda) \ln \left| \operatorname{cth}\left(\frac{\varepsilon(\lambda)}{2T} \right) \right| \right\} \times \left(1 + \mathcal{O}(t^{-\infty})\right)$$

where

$$C(T,h) = \frac{2T\mathcal{A}(T,h)\Phi_p(\lambda_F^-)}{\varepsilon'(\lambda_F^-)}$$

Spacelike asymptotics for fixed T

- THEOREM: [GKS 19] In the spacelike regime $m > 4Jt$ our TFFS for the transversal two-point function is absolutely convergent and determines the late-time long-distance asymptotics

$$t \to +\infty, \quad m \to +\infty \quad \text{at any fixed ratio } \alpha = \frac{m}{4Jt} > 1$$

of the transverse dynamical correlation function of the XX chain explicitly

$$\langle \sigma_1^- \sigma_{m+1}^+(t) \rangle_T = C(T,h)(-1)^m \exp\left\{ -m \int_{\mathcal{C}_h} \frac{\mathrm{d}\lambda}{2\pi} p'(\lambda) \ln \left| \mathrm{cth}\left(\frac{\varepsilon(\lambda)}{2T} \right) \right| \right\} \times \left(1 + \mathcal{O}(t^{-\infty})\right)$$

where

$$C(T,h) = \frac{2T\mathcal{A}(T,h)\Phi_p(\lambda_F^-)}{\varepsilon'(\lambda_F^-)}$$

- MAIN POINT: This is obtained (including the constant $C(T,h)$) directly from the series of multiple integrals by straightforward contour deformations. Such technique should also work in the general XXZ case

Interpretation and low-T behaviour

- Note that the Trotter limit of the first term

$$A_1^{\sigma^-\sigma^+} \times \exp\left\{ -\mathrm{i}ht + \ln(\rho_1(0))m + \frac{\rho_1'(0)}{\rho_1(0)} \cdot \frac{\mathrm{i}t}{\kappa} \right\}$$

of the TFFS gives the same result!

Interpretation and low-T behaviour

- Note that the Trotter limit of the first term

$$A_1^{\sigma^-\sigma^+} \times \exp\left\{ -\mathrm{i}ht + \ln(\rho_1(0))m + \frac{\rho_1'(0)}{\rho_1(0)} \cdot \frac{\mathrm{i}t}{\kappa} \right\}$$

  of the TFFS gives the same result!

- So this expression could generally determine the spacelike asymptotics

Interpretation and low-T behaviour

- Note that the Trotter limit of the first term

$$A_1^{\sigma^-\sigma^+} \times \exp\left\{-iht + \ln(\rho_1(0))m + \frac{\rho_1'(0)}{\rho_1(0)} \cdot \frac{it}{\kappa}\right\}$$

  of the TFFS gives the same result!

- So this expression could generally determine the spacelike asymptotics

- Further note that we have the low-T asymptotic behaviour

$$-\frac{m}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}p \ln\left|\mathrm{cth}\left(\frac{\varepsilon(p)}{2T}\right)\right| \sim -\frac{m\pi T}{2v_F} \qquad (*)$$

  where $v_F = 4J\sin(p_F)$ is the Fermi velocity $\rightarrow$ c.f. Karol's presentation

Interpretation and low-T behaviour

- Note that the Trotter limit of the first term

$$A_1^{\sigma^-\sigma^+} \times \exp\left\{-iht + \ln(\rho_1(0))m + \frac{\rho_1'(0)}{\rho_1(0)} \cdot \frac{it}{\kappa}\right\}$$

  of the TFFS gives the same result!

- So this expression could generally determine the spacelike asymptotics

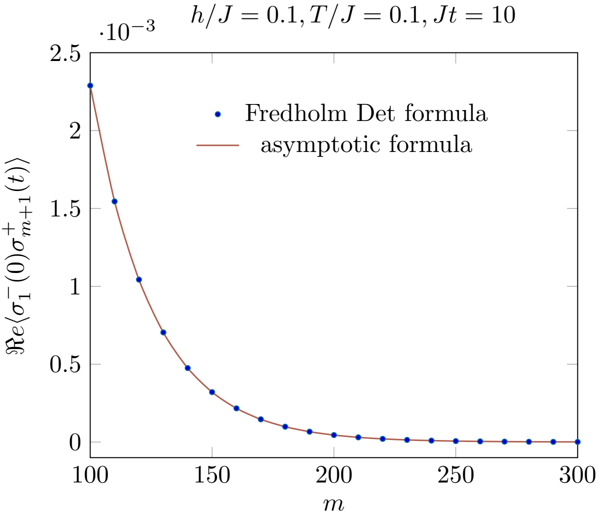
- Further note that we have the low-T asymptotic behaviour

$$-\frac{m}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}p \ln\left|\mathrm{cth}\left(\frac{\varepsilon(p)}{2T}\right)\right| \sim -\frac{m\pi T}{2v_F} \quad (*)$$

  where $v_F = 4J\sin(p_F)$ is the Fermi velocity → c.f. Karol's presentation

- $(*)$ proliferates into the timelike regime and changes at $\alpha_{\min} = \sin(p_F) \;\Leftrightarrow\; m/t = v_F$,
  while at high temperature $\alpha_{\min} = 4J$, the band width

Comparison with numerical result for the full series



Real part of $\langle \sigma_1^-(0)\sigma_{m+1}^+(t)\rangle$ with $T/J = 0.1, h/J = 0.1, Jt = 10$ and $m = 100 \sim 300$ evaluated numerically (dots) and from asymptotic formula

TFFS is different: a Fredholm determinant representation

- THEOREM: [GKSS 19] The transversal correlation functions of the XX chain admit a Fredholm determinant representation of the form

$$\langle \sigma_1^- \sigma_{m+1}^+(t) \rangle_T = (-1)^m \mathcal{F}(m) \Omega(m,t) \det_{\mathcal{C}_p}\big(\mathrm{id} + \widehat{V} - \widehat{P}\big)$$

  where $\widehat{V}$ is an integrable integral operator and P is a one-dimensional projector

- Comparing with the asymptotic behaviour of the correlation function in the spacelike regime $m > 4Jt$ we see that

$$\det_{\mathcal{C}_p}\big(\mathrm{id} + \widehat{V} - \widehat{P}\big) \sim 1 + \mathcal{O}(t^{-\infty})$$

  Fredholm determinant collects the higher-order corrections to the main asymptotics

- [IIKS 93] obtained a different Fredholm determinant representation

$$\langle \sigma_1^- \sigma_{m+1}^+(t) \rangle_T = (-1)^m \Big[\det_{\mathcal{C}}\big(\mathrm{id} + \widehat{W} + \widehat{Q}\big) - \det_{\mathcal{C}}\big(\mathrm{id} + \widehat{W}\big)\Big]$$

  where $\widehat{W}$ is an integrable operator and Q a 1d projector

- We interpret our Fred-det-rep as a ‘resummation’ a la Borodin-Okounkov that is more suitable for the long-time, large-distance asymptotic analysis

Hight-T analysis by means of matrix Riemann-Hilbert problem

- THEOREM: [GKS 20] In the high-T limit the transverse dynamical correlation function of the XX-chain behaves as

$$\langle \sigma_1^- \sigma_{m+1}^+(t) \rangle_T = \frac{1}{2}\left(-\frac{J}{T}\right)^m \exp\left\{ic\tau - \frac{\tau^2}{4} + \int_0^\tau d\tau' \, u_m(\tau')\right\}$$

$$\times \frac{Q_{m+1}(-i)P'_m(-i) - P_m(-i)Q'_{m+1}(-i)}{\left(Q_{m+1}(-i)P'_m(-i) - P_m(-i)Q'_{m+1}(-i)\right)\big|_{\tau=0}}\left(1 + \mathcal{O}(T^{-2})\right)$$

where $\tau = -4J\left(t - \frac{i}{2T}\right)$, $c = \frac{h}{4J}$

$$u_m(\tau) = \frac{i}{2}\left[c_m^{(m-1)} - \gamma_m\left\{F_m(0)\left(P_m(0) - Q'_{m+1}(0)\right) + G_{m+1}(0)P'_m(0)\right\}\right]$$

and were P_m and Q_{m+1} are polynomials with time-dependent coefficients obeying fully explicit linear equations

Hight-T analysis by means of matrix Riemann-Hilbert problem

- THEOREM: [GKS 20] In the high-T limit the transverse dynamical correlation function of the XX-chain behaves as

$$\langle \sigma_1^- \sigma_{m+1}^+(t) \rangle_T = \frac{1}{2}\left(-\frac{J}{T}\right)^m \exp\left\{ \mathrm{i}c\tau - \frac{\tau^2}{4} + \int_0^\tau \mathrm{d}\tau' \, u_m(\tau') \right\}$$
$$\times \frac{Q_{m+1}(-\mathrm{i})P'_m(-\mathrm{i}) - P_m(-\mathrm{i})Q'_{m+1}(-\mathrm{i})}{\left(Q_{m+1}(-\mathrm{i})P'_m(-\mathrm{i}) - P_m(-\mathrm{i})Q'_{m+1}(-\mathrm{i})\right)\big|_{\tau=0}} \left(1 + \mathcal{O}(T^{-2})\right)$$

where $\tau = -4J\left(t - \frac{\mathrm{i}}{2T}\right)$, $c = \frac{h}{4J}$

$$u_m(\tau) = \frac{\mathrm{i}}{2}\left[c_m^{(m-1)} - \gamma_m\left\{F_m(0)(P_m(0) - Q'_{m+1}(0)) + G_{m+1}(0)P'_m(0)\right\}\right]$$

and were P_m and Q_{m+1} are polynomials with time-dependent coefficients obeying fully explicit linear equations

- COROLLARY: (Generalizing [BJ 77]) In the high-T limit the transverse auto-correlation function of the XX-chain behaves as

$$\langle \sigma_1^- \sigma_1^+(t) \rangle_T = \frac{1}{2} \mathrm{e}^{-\mathrm{i}h(t-\mathrm{i}/(2T)) - 4J^2(t-\mathrm{i}/(2T))^2} \left(1 + \mathcal{O}(T^{-2})\right)$$