**INTEGRABLE SYSTEMS AND FIELD THEORY**

65-th Birthday of Fedor Smirnov

*11-13 oct. 2023 Paris (France)*

# Spectral properties of planar CFT's in D>2: from N=4 SYM to Fishnet CFT

Vladimir Kazakov

# A bit of history…

- I met Fedya many years ago in Russia, but we started our scientific discussions (and became friends) only in Paris, in the 90s

- I learned a lot from from him on quantum and classical integrability (in which I was a beginner at the time…)

F.Smirnov '98
Babelon, Bernanrd, F.Smirnov '97

- Somewhen around 2003, he explained me the finite gap method on the example of his work on KdV field theory. It was crucial for my contribution to one of the decisive works towards complete solution of spectral problem of planar $\mathcal{N}$=4 SYM

V.K., Marshakov, Minahan, Zarembo '04

## Plan of the talk

Gromov, V.K., Leurent, Volin '13, '14
V.K., Leurent, Volin '15

- Quantum Spectral Curve (QSC) for planar spectrum of anomalous dimensions in $\mathcal{N}$=4 SYM

- Fishnet CFT as a weak coupling of strongly $\gamma$-deformed $\mathcal{N}$=4 SYM — Gurdogan, V.K., '15

- Generalized "Loom" Fishnet CFT: construction, examples, Yangian symmetry

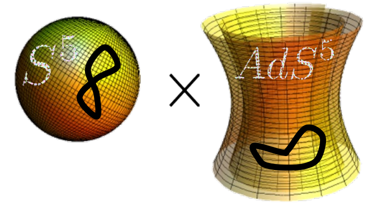
V.K., Olivucci '22
V.K., Levkovich-Maslyuk, Mishnyakov '23
Alfimov, V.K., Ferrando, Olivucci (in preparation)

# Integrability of planar $\mathcal{N}$=4 SYM

# $\mathcal{N}$=4 SYM: planar integrability from AdS/CFT duality

$$\mathcal{S}_{SYM} = \frac{1}{\lambda}\int d^4x \,\mathrm{Tr}\left(F^2 + (\mathcal{D}\Phi)^2 + [\Phi,\Phi]^2\right) + \text{fermions}$$

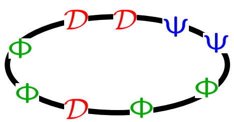
dual to




Maldacena '97
Gubser,Klebanov,Polyakov
Witten

super-conformal theory: PSU(2,2|4) symmetry
β-function=0, no massive particles

operators




$$\mathcal{O}(x) = \mathrm{Tr}\,[\mathcal{D}\mathcal{D}\Psi\Psi\Phi\Phi\mathcal{D}\Psi\ldots]\,(x)$$

Anomalous dimensions $\Delta_{\mathcal{O}}$

$$\mathcal{O}(\xi x) \to \xi^{\Delta_{\mathcal{O}}(\lambda)}\mathcal{O}(x)$$

Operator product expansion,
structure constants, correlators, amplitudes…

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_k |x|^{-\Delta_i-\Delta_j+\Delta_k}\, C^k_{i,j}\, \mathcal{O}_k(0) + \ldots$$

1- and 2-loop integrability
Classical integrability
S-matrix, asymptotic Bethe ansatz

**Bena, Roiban, Polchinski 02'**
**Metsaev-Tseytlin 02'**
**Minahan, Zarembo 03'**
**Beisert, Kristjansen, Staudacher 03'**
**V.K.,Marshakov, Minahan, Zarembo 04'**
**Beisert, V.K.,Sakai, Zarembo 05'**
**Beisert,Staudacher 04'**
**Beisert 05'**
**Janik 05'**
**Beisert, Eden, Staudacher '06**

→ Y-system

**Gromov, V.K,, Vieira '09**

→ Thermodynamical Bethe ansatz (TBA)

**Bombardelli,Fioravanti,Tateo '09**
**Gromov,V.K.,Kozak,Vieira '09**
**Arutyunov,Frolov '09**
**Cavaglia, Fioravanti, Tateo '09**

→ Quantum spectral curve (QSC)

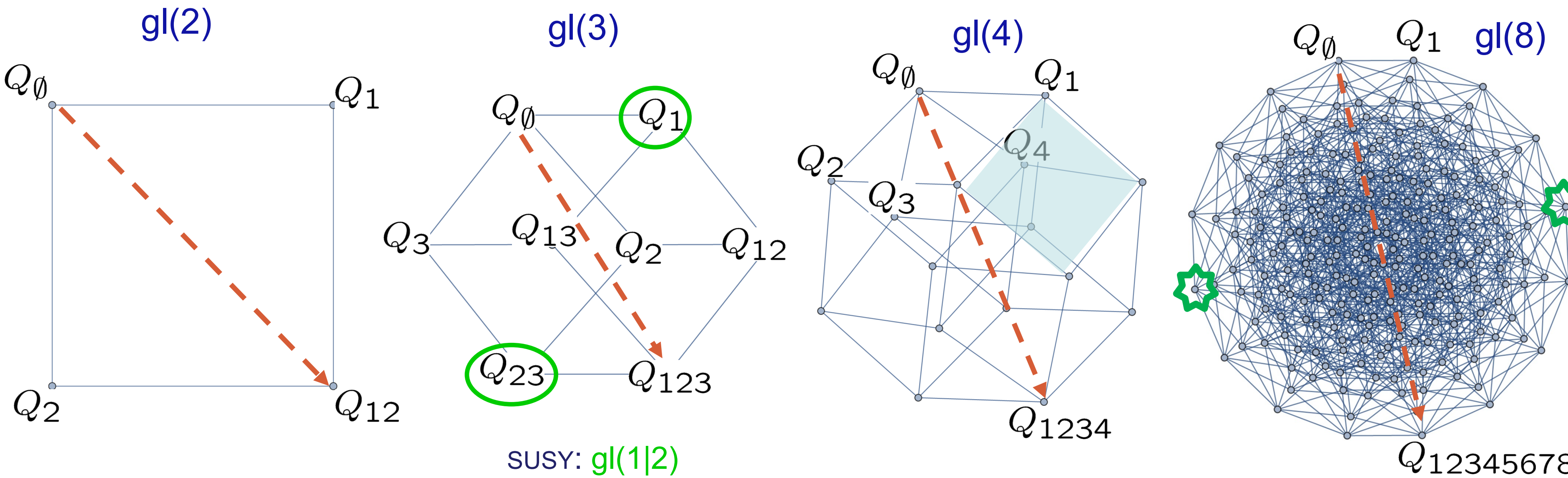
**Gromov, V.K., Leurent, Volin '13, '14**
**V.K., Leurent, Volin '15**

# Quantum Spectral Curve of $\mathcal{N}$=4 SYM: algebraic structure

Gromov, V.K., Leurent, Volin '13,'14
V.K., Leurent, Volin '15

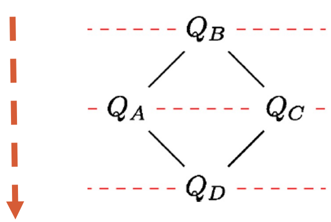
- QSC eqs. close on finite set of Baxter functions of spectral parameter $Q_I(u)$
- gl(n): each Q placed at an edge of Hasse diagram - n-hypercube

for $\mathcal{N}$=4 SYM




SUSY: gl(8) → gl(4|4)

- Plücker QQ-relations on each face ("determinant flow"):




$$\Leftrightarrow \quad Q_B(u)Q_D(u) = \begin{vmatrix} Q_A(u+\frac{i}{2}) & Q_C(u+\frac{i}{2}) \\ Q_A(u-\frac{i}{2}) & Q_C(u-\frac{i}{2}) \end{vmatrix}$$

Grassmanian structure!

Krichever, Lupan, Wiegmann, Zabrodin, '94
Tsuboi '13

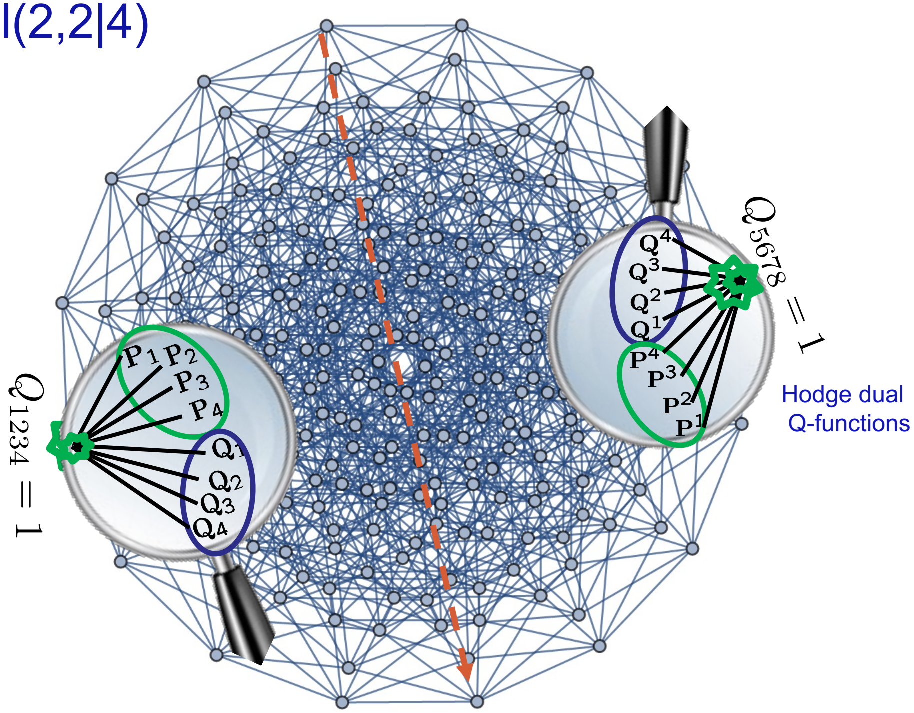
- gl(N) Heisenberg spin chain: $Q_0 = 1, \qquad Q_j = \text{Polynomial}(u), \qquad Q_{1,2,\ldots,N} \sim u^{\text{Length}}$
- gl(M|K) Heisenberg spin chain: $Q_{12\ldots M} = 1, \qquad Q_{\text{neighbors}\,12\ldots M} = \text{Polyn}(u) \qquad Q_{M+1,M+2,\ldots K} \sim u^{\text{Length}}$

Gromov, V.K., Leurent, Volin '13,'14
V.K., Leurent, Volin '15

# Quantum Spectral Curve of $AdS_5/CFT_5$: analytic structure

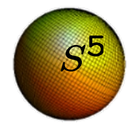
- Special 8 + 8 Q-functions with nice analyticity on physical sheet

gl(2,2|4)



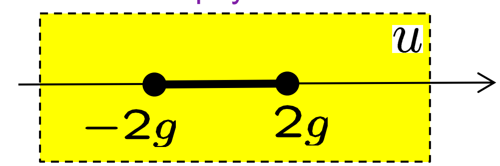

Large u asymptotics fixed by PSU(2,2|4)

Cartan charges $\{\Delta, S_1, S_2 | J_1, J_2, J_3\}$ (SU(2,2) | SU(4))




$$\mathbf{P}_b,\ \mathbf{P}^b \sim q_b^{\pm iu}\, u^{(\pm J_1 \pm J_2 \pm J_3)/2}$$

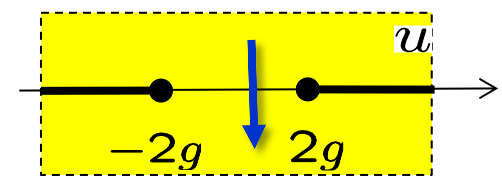
short cut on physical sheet





$$\mathbf{Q}_j,\ \mathbf{Q}^j \sim u^{(\pm\Delta \pm S_1 \pm S_2)/2}$$

long cut on physical sheet




- Various Q-functions are related by complex conjugation ("gluing" relations)

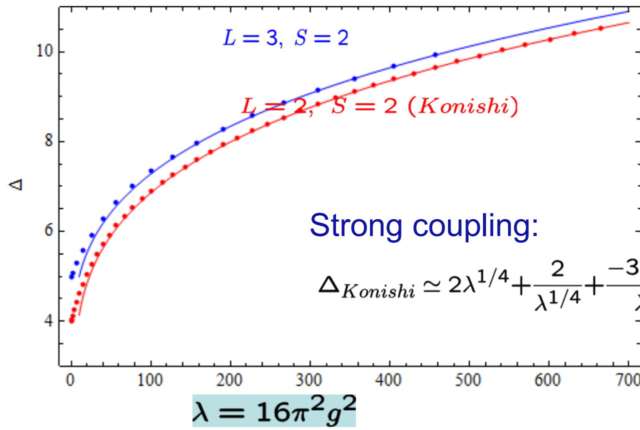
$$\mathbf{Q}_1 \propto \bar{\mathbf{Q}}^2, \quad \mathbf{Q}_2 \propto \bar{\mathbf{Q}}^1, \quad \mathbf{Q}_3 \propto \bar{\mathbf{Q}}^4, \quad \mathbf{Q}_4 \propto \bar{\mathbf{Q}}^3$$

- These Riemann-Hilbert conditions fix all physical solutions for Q-system and thus conformal dimensions $\Delta(g)$ with given PSU(2,2|4) charges

# Dimensions of twist-2,3,… operators $\mathrm{Tr}\,(\Phi\,\nabla^S\,\Phi)$

- Numerics, weak and strong coupling from Quantum Spectral Curve;

Gromov,V.K.,Vieira ‘09
Frolov ‘10
Gromov,Valatka ‘12



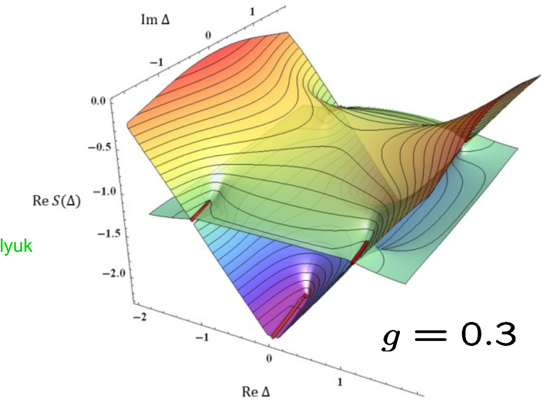

Strong coupling:

$$\Delta_{Konishi} \simeq 2\lambda^{1/4} + \frac{2}{\lambda^{1/4}} + \frac{-3\zeta_3+\frac{1}{2}}{\lambda^{3/4}} + \frac{\frac{15}{2}\zeta_5+6\zeta_3-\frac{1}{2}}{\lambda^{5/4}} + \dots$$

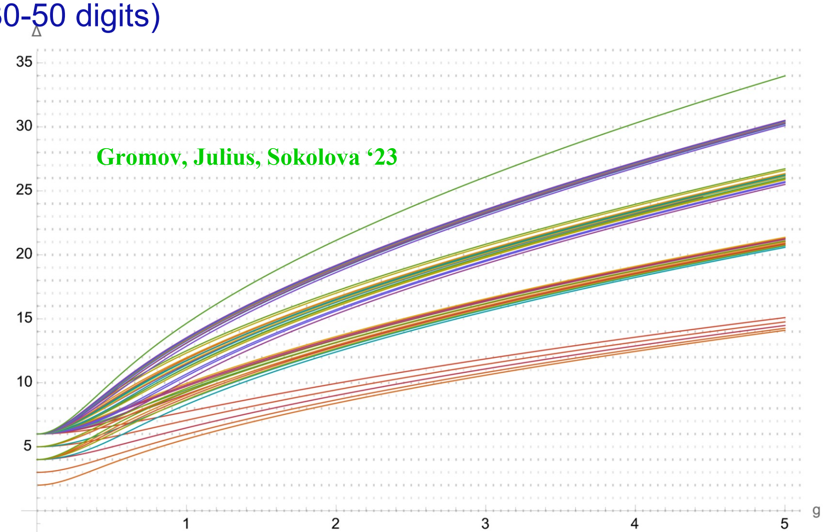
Gromov, Valatka, Sizov, Levkovich-Maslyuk
Gromov,Shenderovich,
Serban, Volin
Roiban, Tseytlin
Vallilo, Mazzucato
Gubser, Klebanov, Polyakov

Function of complex conformal spin $\Delta(S,g)$

Gromov, Levkovich-Maslyuk, Sizov ‘15




Recent results “unlimited” precision
(~30-50 digits)




Weak couping (11 loops)

Gromov,VK,Leurent, Volin ‘13
Leurent, Serban, Volin ‘12
Volin, Marboe ‘18

$$\gamma_{Konishi} = \sum_{j=1}^{\infty} g^{2j}\,\gamma_j$$

$$\begin{aligned}\gamma_{11} = {} & -242508705792 + 107663966208\zeta_3 + 70251466752\zeta_3^2 - 12468142080\zeta_3^3 \\ & +1463132160\zeta_3^4 - 71663616\zeta_3^5 + 180173002752\zeta_5 - 16655486976\zeta_3\zeta_5 \\ & -24628230144\zeta_3^2\zeta_5 - 2895575040\zeta_3^3\zeta_5 + 19278176256\zeta_5^2 - 9619845120\zeta_3\zeta_5^2 \\ & +2504494080\zeta_3^2\zeta_5^2 + \tfrac{882108048384}{175}\zeta_5^3 + 45602231040\zeta_7 + 14993482752\zeta_3\zeta_7 \\ & -12034759680\zeta_3^2\zeta_7 + 1406730240\zeta_3^3\zeta_7 + 30605033088\zeta_5\zeta_7 + 21217637376\zeta_3\zeta_5\zeta_7 \\ & -\tfrac{1309941061632}{275}\zeta_5^2\zeta_7 - 13215327552\zeta_7^2 - 4059901440\zeta_3\zeta_7^2 - 69762034944\zeta_9 \\ & +23284599552\zeta_3\zeta_9 - 3631889664\zeta_3^2\zeta_9 - 11032374528\zeta_5\zeta_9 - 6666706944\zeta_3\zeta_5\zeta_9 \\ & -23148129024\zeta_7\zeta_9 - 10024051968\zeta_9^2 - 54555179184\zeta_{11} + \tfrac{10048541184}{5}\zeta_3\zeta_{11} \\ & -726029568\zeta_3^2\zeta_{11} - 8975463552\zeta_5\zeta_{11} - 22529041920\zeta_7\zeta_{11} - \tfrac{14379934224 96}{175}\zeta_{13} \\ & +\tfrac{1504385419392}{35}\zeta_3\zeta_{13} - 30324602880\zeta_5\zeta_{13} - \tfrac{151130039581392}{875}\zeta_{15} - 41375093760\zeta_3\zeta_{15} \\ & -\tfrac{196484147423712}{275}\zeta_{17} + 309361358592\zeta_{19} - 1729880064Z_{11}^{(2)} - \tfrac{1620393984}{5}\zeta_3 Z_{11}^{(2)} \\ & -131383296\zeta_5 Z_{11}^{(2)} + \tfrac{138107420928}{175}Z_{13}^{(2)} + \tfrac{3543865344}{35}\zeta_3 Z_{13}^{(2)} - \tfrac{5716780416}{7}Z_{13}^{(3)} \\ & -\tfrac{674832384}{7}\zeta_3 Z_{13}^{(3)} + \tfrac{48227088384}{175}Z_{15}^{(2)} + \tfrac{3581880576}{25}Z_{15}^{(3)} + 754974720 Z_{15}^{(4)} \\ & -\tfrac{854924544}{11}Z_{17}^{(2)} + \tfrac{4963244544}{55}Z_{17}^{(3)} + \tfrac{818159616}{275}Z_{17}^{(4)} + \tfrac{175363688448}{1925}Z_{17}^{(5)}\,. \end{aligned}$$
(A.

- QSC for ABJM, AdS3/CFT2

Cavaglia, Fioravanti,Gromov,Tateo ‘14
Cavaglia, Gromov, Stefanski,Torielli ‘21

# Fishnet CFT

# γ-twisted N=4 SYM and "fishnet" limit

Gurdogan, V.K. '15

$\phi_1$ $\phi_2$ $\phi_3$ $\psi_1$ $\psi_2$ $\psi_3$ $\lambda$ $A$

$$\mathcal{L} = N_c \mathrm{tr}\Big(F^2 + D\bar{\phi}_i D\phi_i + i\bar{\psi}_j \not{D}\psi_j + i\bar{\lambda}\not{D}\lambda +$$
$$+ g^2 [\phi_j, \phi_k]_q \cdot [\bar{\phi}_j, \bar{\phi}_k]_q + i\, g\, \epsilon_{ijk} \bar{\psi}_k [\phi_i, \bar{\psi}_j]_q + g\, \bar{\lambda}[\phi_j, \bar{\psi}_j]_q + \text{ conj.}\Big)$$

- γ-twisted N=4 SYM Lagrangian: commutators → q-commutators

$$[A, B] \to [A, B]_q \equiv q_{AB}\, A\, B - \frac{1}{q_{AB}} B\, A \qquad \text{where} \quad q_{A,B} = e^{-\frac{i}{2}\epsilon^{mjk}\gamma_m J_j^A J_k^B} = (q_{B,A})^{-1}$$

$J_1^A,\ J_2^A,\ J_3^A \in SO(6)$ - Cartan charges of R-symmetry

$\gamma_1, \gamma_2, \gamma_3$ - twists

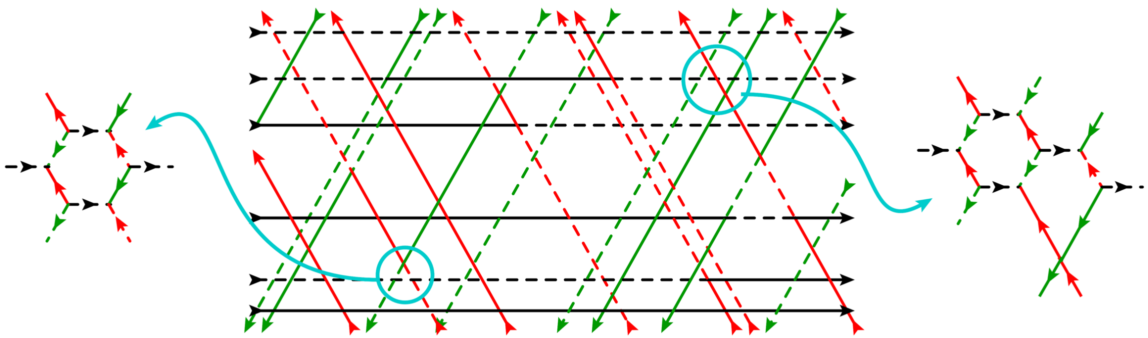
Leigh, Strassler
Frolov, Tseytlin
Beisert, Roiban
Lunin, Maldacena

- Breaks R-symmetry and all supersymmetry: PSU(2,2|4)→SU(2,2)×U(1)$^3$
- Double scaling "fishnet" limit: Strong imaginary twist, weak coupling:

Gurdogan, V.K. '15

$$g \to 0, \qquad \gamma_j \to i\infty, \qquad \xi_j = g\, e^{-i\gamma_j/2} - \text{fixed}, \qquad (j = 1, 2, 3.)$$

- Planar Feynman graphs form a dynamical fishnet: solid lines – bosons, dashed lines - fermions



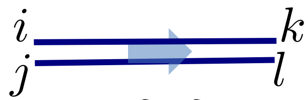

- Integrable, since deduced from integrable N=4 SYM! But how to see it explicitly?

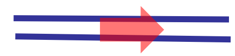
# Special case: bi-scalar Fishnet CFT$_4$

Gurdogan, V.K. 2015

- Retain only one coupling in fishnet limit of N=4 SYM: $\xi_1 = \xi_2 = 0\,, \quad \xi_3 \equiv \xi \neq 0$

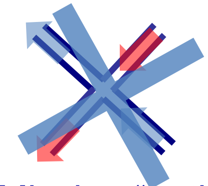
$$\mathcal{L}[\phi_1,\phi_2] = \frac{N_c}{2}\mathrm{tr}\,\left(\partial^\mu \bar\phi_1 \partial_\mu \phi_1 + \partial^\mu \bar\phi_2 \partial_\mu \phi_2 + 2\xi^2\,\bar\phi_1\bar\phi_2\phi_1\phi_2\right)$$



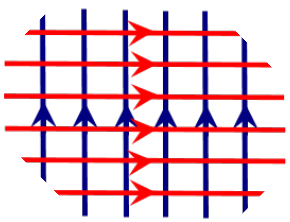
propagators $= \dfrac{\delta_{ik}\,\delta_{jl}}{N_c\,(x-y)^2}$

vertex



Missing "anti-chiral" vertex

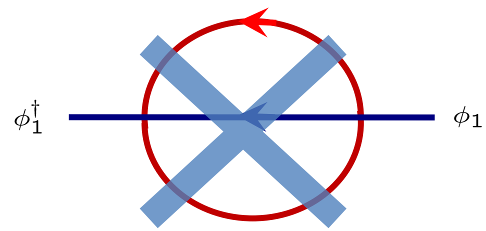
- $N$ = 4 SYM planar graphs reduce, in the bulk, to (very few!) fishnet graphs.

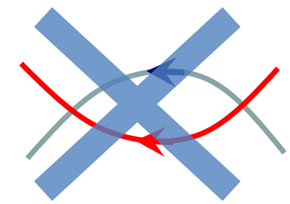


Integrable!

A.Zamolodchikov '80

0-dim analogue:
Kostov, Staudacher '96



- No mass or vertex renormalization! Coupling $\xi$ does not run!

- But there are double-trace counterterms with $\xi$ dependent couplings:

$$\mathrm{tr}\left(\bar\phi_1\bar\phi_2\right)\,\mathrm{tr}\left(\bar\phi_1\phi_2\right)\,, \qquad \mathrm{tr}\left(\phi_1\bar\phi_2\right)\,\mathrm{tr}\left(\bar\phi_1\phi_2\right))\,, \qquad \mathrm{tr}\left(\phi_j\bar\phi_j\right)\,\mathrm{tr}\left(\phi_j\bar\phi_j\right)$$

Sieg, Wilhelm '16
Grabner, Gromov, V.K.,Korchemsky, '17

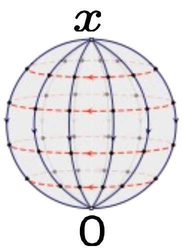
- One can study correlators of local operators

$$\mathcal{O}(x) = C^{\mu_1\cdots\mu_n}\,\mathrm{tr}\left[\partial_{\mu_1}\ldots\partial_{\mu_n}(\phi_1)^L(\phi_2)^M(\bar\phi_1)^K(\bar\phi_2)^N\right](x) + \text{permutations}$$

# Operators, correlators, graphs…

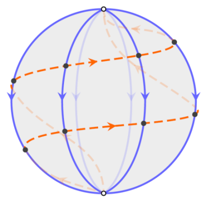
- Explicit computations of correlators

"vacuum" operator $\mathrm{tr}[\phi_1(x)]^L$

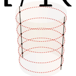



Gurdogan, V.K '15
Caetano, Gurdogan, V.K '16

Multi-magnon spiral graphs



$\mathrm{tr}[\phi_1(x_1) \quad \phi_1(x_2)]$



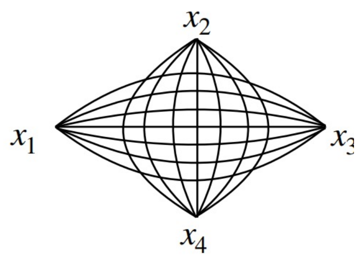
$\mathrm{tr}[\phi_1^\dagger(x_3) \quad \phi_1^\dagger(x_4)]$

OPE, 4-point functions, stampedes…

Grabner, Gromov, V.K , Korchemsky '17
V.K., Olivucci 2018
Gromov, V.K , Korchemsky '18
V.K., Olivucci, Preti, '19
Pittelli, Preti '19
Gromov, Sever '20
Olivucci, Vieira '22
Olivucci '23

- Basso-Dixon 4-point functions through determinant of "ladder" graphs, Sklyanin SoV

$$G_{m,n}(x_1, x_2, x_3, x_4) \; = $$

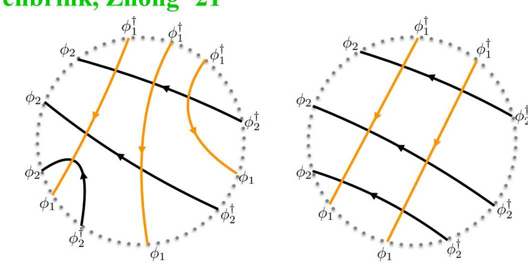



Davidichev, Ushuikina
Basso, Dixon
Derkachev, V.K., Olivucci
Derkachev, Ferrando, Olivucci '21
Dercachov, Olivucci '21
Basso, Dixon, Kosover, Krajenbrink, Zhong '21
Kostov '23
…



- Amplitudes, Yangian symmetry, Calabi-Yau periods…

Chicherin, V.K., Mueller, Loebbert, Zheng '17
Corcoran, Loebbert, Miczajka, Muller, Munkler '20
Duhr, Klemm Loebbert, Nega, Porkert'22
V.K., Levkovich-Maslyuk, Mishnyakov '23



- Thermodynamical Bethe Ansatz for Fishnet

Basso, Zhong '19
Basso, Ferrando, V.K., Zhong '19

- "Fishchain": AdS dual for Fishnet

Gromov, Sever '19

- Non-trivial flat vacua in Fishnet CFT

V.K., Karananas, Shaposhnikov '19

- Eclectic spin chain from Fishnet

Ipsen, Staudacher, Zippelius '18
Ahn, Staudacher '21, '22

- Beyond Fishnet limit of N=4 SYM

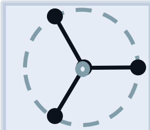
Ferrando, Sever '23

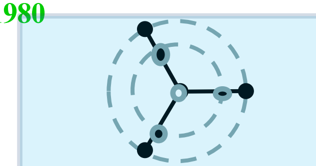
- Thermodynamics of Fishnet graphs

Zamolodchikov '80
Staudacher, Kade '23

# Dimension of $\mathtt{tr}(\phi_1)^3$ and periods of wheel graphs from QSC

Broadherst 1980

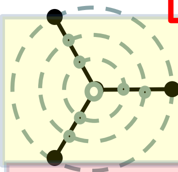


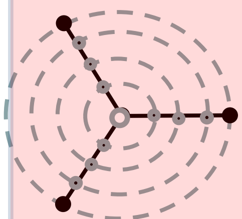


Ahn, Bajnok, Bombardelli, Nepomechie 2013

E.Panzer, 2015 Gurdogan, V.K. ’15 (any number of spokes)

In terms of Riemann (multi)-zeta numbers





Gromov, V.K , Korchemsky, Negro, Sizov ’17

$$\Delta - 3 = -12\xi^6\zeta(3) + \xi^{12}\left(189\zeta(7) - 144\zeta(3)^2\right)$$
$$+\xi^{18}\left(-1944\zeta(8,2,1) - 3024\zeta(3)^3 - 3024\zeta(5)\zeta(3)^2 + \frac{198\pi^8\zeta(3)}{175} + 6804\zeta(7)\zeta(3) + \frac{612\pi^6\zeta(5)}{35} + 270\pi^4\zeta(7) + 5994\pi^2\zeta(9) - \frac{925911\zeta(11)}{8}\right) +$$
$$\xi^{24}\left(\frac{10368}{5}\pi^4\zeta(8,2,1) + 5184\pi^2\zeta(9,3,1) + 51840\pi^2\zeta(10,2,1) - 148716\zeta(11,3,1) - 1061910\zeta(12,2,1) + 62208\zeta(10,2,1,1,1) - 93312\zeta(3)\zeta(8,2,1) - 288\zeta(3)^5 + 72\gamma\pi^2\zeta(3)^4 - 77760\zeta(3)^4 - \frac{80756\pi^6\zeta(3)^3}{945} - 145152\zeta(5)\zeta(3)^3 - \frac{29}{270}\gamma\pi^8\zeta(3)^2 + \frac{9504\pi^8\zeta(3)^2}{175} - 879\pi^4\zeta(5)\zeta(3)^2 - 2025\pi^2\zeta(7)\zeta(3)^2 + 244944\zeta(7)\zeta(3)^2 + 186588\zeta(9)\zeta(3)^2 + \frac{2910394\pi^{12}\zeta(3)}{2627625} - 2592\pi^2\zeta(5)^2\zeta(3) + \frac{29376}{35}\pi^6\zeta(5)\zeta(3) + 12960\pi^4\zeta(7)\zeta(3) + 298404\zeta(5)\zeta(7)\zeta(3) + 287712\pi^2\zeta(9)\zeta(3) - 5555466\zeta(11)\zeta(3) + 57672\zeta(5)^3 - 71442\zeta(7)^2 + \frac{13953\pi^{10}\zeta(5)}{1925} + \frac{7293\pi^8\zeta(7)}{175} - \frac{19959\pi^6\zeta(9)}{5} + \frac{119979\pi^4\zeta(11)}{2} + \frac{10738413\pi^2\zeta(13)}{2} - \frac{4607294013\zeta(15)}{80}\right) + O\left(\xi^{25}\right)$$

- Based on Quantum Spectral curve of $\mathcal{N}$=4 SYM

Baxter eq.:
$$\left(\frac{(\Delta-1)(\Delta-3)}{4u^2} - \frac{i\,\xi^3}{u^3} - 2\right) q(u) + q(u+i) + q(u-i) = 0$$

Asymptotics:
$$q_1(u,\xi) \sim u^{\Delta/2-1/2}\left(1 + \frac{\alpha_1}{u} + \frac{\alpha_2}{u^2} + \cdots\right)$$
$$q_2(u,\xi) \sim u^{-\Delta/2+3/2}\left(1 + \frac{\beta_1}{u} + \frac{\beta_2}{u^2} + \cdots\right)$$

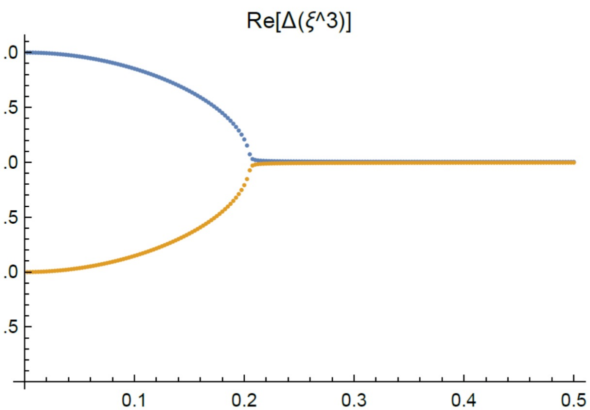
Quantization condition
$$q_1(0,\xi)\, q_2(0,-\xi) + q_2(0,-\xi)\, q_1(0,\xi) = 0$$

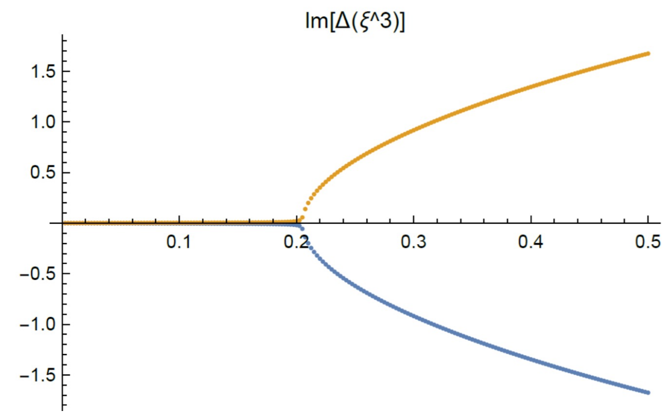
Interesting relation to Galois coaction on Feynman periods Gurdogan‘21

# High precision numericsfor spectrum and “PT” symmetry

Gromov, V.K , Korchemsky, Negro, Sizov ‘17

$\mathrm{tr}(\phi_1)^3$




- The two dimensions are real for $\xi < \xi_c$ , but they turn to complex conjugates for $\xi > \xi_c$
- The reason for this *reality of spectrum*: “PT” symmetry of Fishnet CFT V.K., Olivucci ‘22
  “PT”-transformation leaves the action invariant (but not operators!):

complex conjugate — transpose

$$\mathrm{tr}(\phi_1\phi_2\bar\phi_1\bar\phi_2) \xrightarrow{T} \mathrm{tr}(\phi_2\phi_1\bar\phi_2\bar\phi_1) \xrightarrow{\text{“}P\text{”}} \mathrm{tr}(\phi_1\phi_2\bar\phi_1\bar\phi_2)$$

Conformal dimension gets complex conjugate (non-unitary theory!):

$$\left[\langle\bar{\mathcal{O}}(x)\mathcal{O}(0)\rangle\right]^{\mathrm{PT}} = \langle\bar{\mathcal{O}}^{\mathrm{PT}}(x)\mathcal{O}^{\mathrm{PT}}(0)\rangle = |x|^{-2\Delta^*}$$

The spectrum consists of real dimensions and/or of complex conjugate pairs!

Similar to energy spectrum of non-unitary PT-invariant quantum mechanics

$$\mathcal{H} = \hat p^2/2 + x^2\,(ix)^\epsilon$$

Bender & Boettcher ‘98

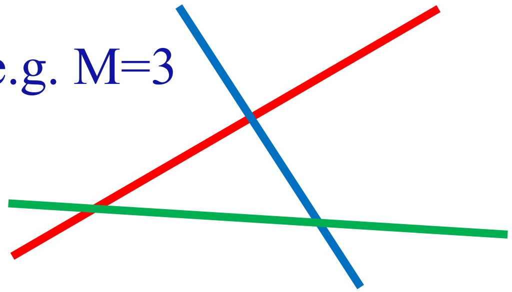
V.K., Olivucci '22

# Loom for fishnet CFTs from Baxter lattices

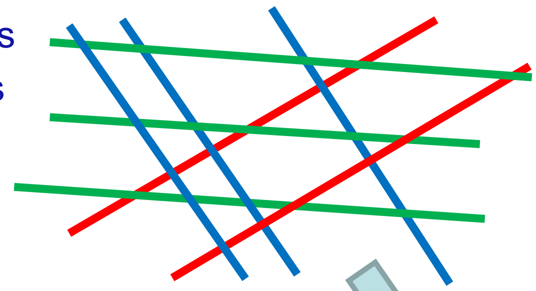
Based on: A.Zamolodchikov 1980

- Baxter lattice for general Fishnet CFT: M intersecting lines with M slopes

e.g. M=3

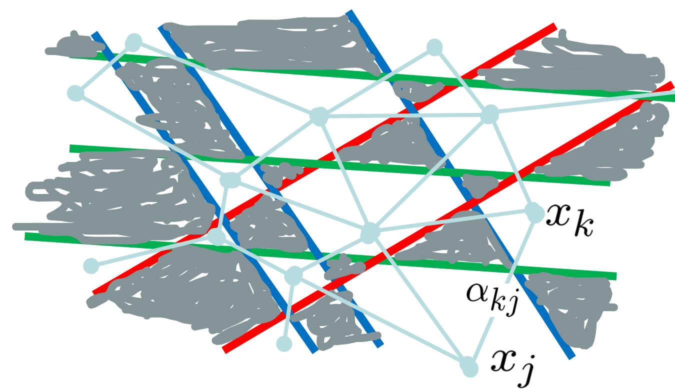


add parallel lines at any positions (loom)



Color "odd" faces in the "Checkerboard" manner

Connect vertices at even (white) faces by appropriate propagators

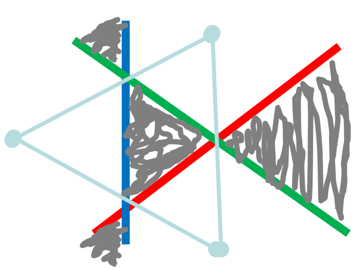


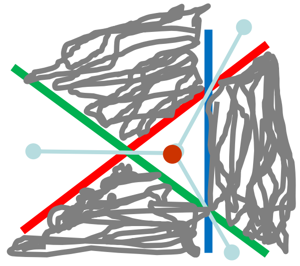

Feynman integral for this graph:

$$\mathcal{G}_B = \int \prod_{m\in\mathcal{L}_I} d^D x_m \prod_{<j,k>\in\mathcal{L}_I} G_D(x_j, x_k, \alpha_{jk})$$

$$G_D(x_j, x_k, \alpha_{jk}) = |x_j - x_k|^{\frac{D}{\pi}(\alpha_{jk}-\pi)}$$

Integrability of such "loom" graphs based on star-triangle relation



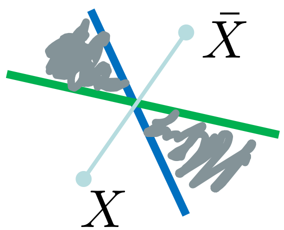


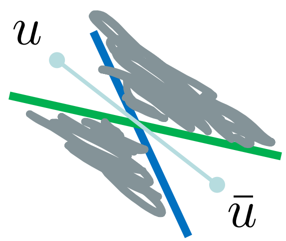
All Feynman graphs of the loom are connected by star-triangle relations!

We will construct Fishnet CFT$^{(M)}$ with all such Feynman graphs

# Generalized Fishnet CFT: Kinetic Terms

To accommodate all these graphs within a Fishnet CFT$^{(M)}$, we need $M(M-1)$ scalar fields (two for each crossing)




Kinetic terms are defined by dimensions of fields:

$$\mathcal{L}_{\rm kin} = \frac{N_c}{2} \mathrm{tr} \left( - \sum_{j=1}^{M(M-1)} \bar{\phi}_i \left( \Box^{D/2-\Delta_{\phi_i}} \right) \phi_i \right)$$

Example of $M=3$:
6 fields with dimensions

$$\Delta_X$$

$$\Delta_Y$$

$$\Delta_Z = \frac{D}{2} - \Delta_X - \Delta_Y$$

$$\Delta_u = \frac{D}{2} - \Delta_X$$

$$\Delta_v = \frac{D}{2} - \Delta_Y$$

$$\Delta_w = \Delta_X + \Delta_Y$$

V.K., Olivucci '22

# Generalized Fishnet CFT: Interactions

Construct the vertices: start from the largest one and consecutively replace pairs of fields by the "dual" fields (according to star-triangle):

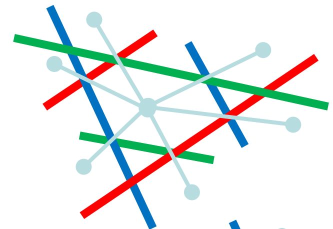
$XY \to w$

$YZ \to u$

$Z\bar{X} \to v$

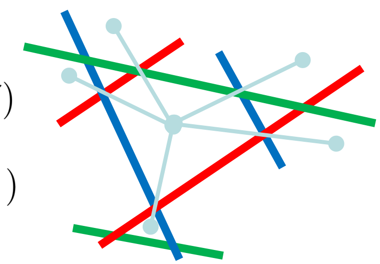
For example, general FCFT[(3)] has 18 vertices:

1 sextic  tr $(XYZ\bar{X}\bar{Y}\bar{Z})$



6 quintic  tr $(w\,Z\bar{X}\bar{Y}\bar{Z})$  tr $(X\,u\,\bar{X}\bar{Y}\bar{Z})$  tr $(XY\,v\,\bar{Y}\bar{Z})$

tr $(XYZ\,\bar{w}\,\bar{Z})$  tr $(XYZ\bar{X}\,\bar{u}\,)$  tr $(YZ\bar{X}\bar{Y}\,\bar{v}\,)$

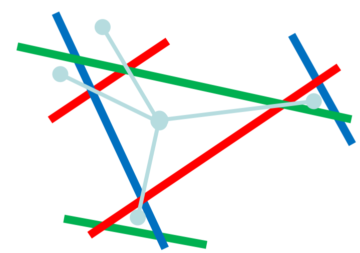


Each vertex has its independent coupling $\xi_j$

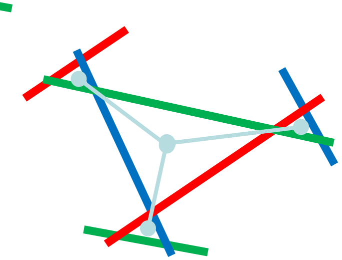
9 quartic  tr $(w\,v\,\bar{Y}\bar{Z})$  tr $(w\,Z\bar{X}\,\bar{u})$  tr $(XY\,v\,\bar{u})$

tr $(u\,\bar{X}\bar{Y}\,\bar{v})$  tr $(XY\,v\,\bar{u})$  tr $(YZ\,\bar{w}\bar{v})$

tr $(Y\,v\,\bar{Y}\,\bar{v})$  tr $(w\,Z\,\bar{w}\,\bar{Z})$  tr $(u\,Z\bar{X}\,\bar{u}\,Z)$



2 cubic  tr $(v\,\bar{w}\,\bar{u})$  tr $(w\,v\,\bar{u})$



There are also double-trace terms…

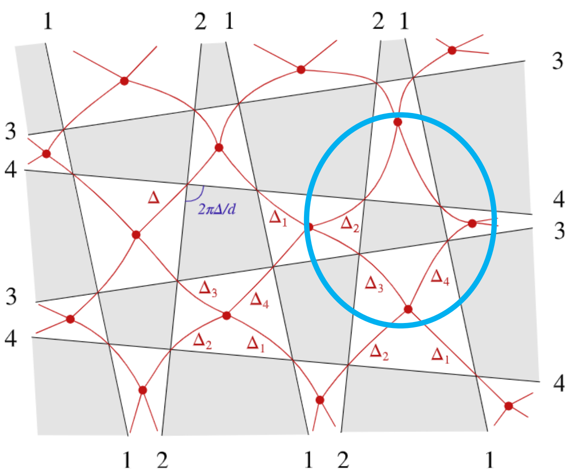
These "Loom" Fishnet CFTs have integrable planar graphs at any D !

Alfimov, V.K., Ferrando, Olivucci, to appear

# Checkerboard FCFT$^{(4)}$

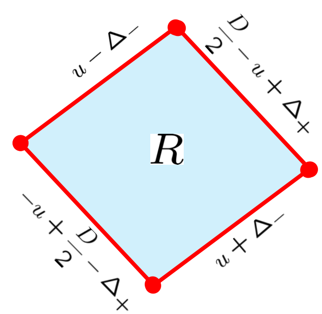
A Loom FCFT with M=4 slopes but only two interaction terms turned on:

$$\mathcal{L}_D^{(CD)}\,|_{w_4=0} = N_c \mathrm{tr}\left[-\sum_{j=1}^{4} \bar{X}_j \Box^{w_j} X_j + \xi_1^2\,\bar{X}_1\bar{X}_2 X_3 X_4 + \xi_2^2\, X_1 X_2 \bar{X}_3 \bar{X}_4\right], \qquad \sum_{j=1}^{4} w_j = D$$




R-matrix in principals series irrep of conformal group

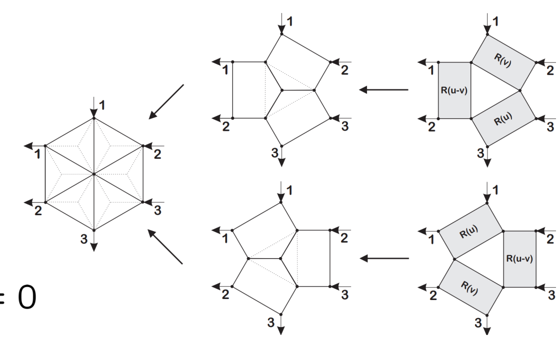
Derkachov, Korchemsky, Manashov ‘01
Chicherin, Derkachev, Isaev’12





$= |x_1 - x_2|^{-2a}$

Satisfies Yang-Baxter relation:



- Local action in 4D $\quad w_1 = w_2 = w_3 = w_4 = 1$

- ABJM FCFT$^{(3)}$ emerges in 3D, for $\quad w_1 = w_2 = w_3 = 1, \quad w_4 = 0$

$$\mathcal{L}_3^{(\mathrm{ABJM\ FCFT})} = N_c \mathrm{tr}\left[\sum_{j=1}^{3} \partial^\mu \bar{X}_j \partial_\mu X_j + (\xi_1\xi_2)^2\,\bar{X}_1\bar{X}_2 X_3 X_1 X_2 \bar{X}_3\right]$$

Caetano, Gurdogan, V.K., ‘16

- At $u \to 0$ for $\quad D = 2, \quad w_1 = u+2,\ w_2 = -u,\ w_3 = u,\ w_4 = -u$
  transfer matrix becomes Lipatov’s Hamiltonian for reggeized gluons in QCD

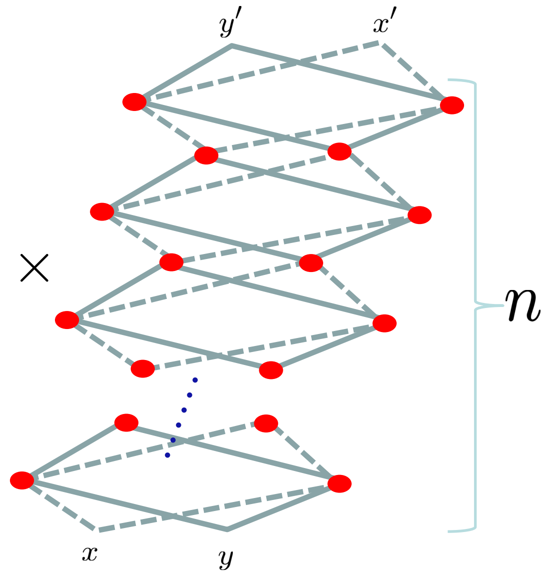
Alfimov, V.K., Ferrando, Olivucci, to appear

# 4-point correlator for Checkerboard FCFT$^{(4)}$

$$\langle \mathrm{tr}[X_1X_2(x)\ X_1X_2(y)]\ \mathrm{tr}[\bar{X}_3\bar{X}_4(x')\ \bar{X}_3\bar{X}_4(y')]\rangle_c$$

- Given by ladder graphs (each square is R-matrix)

$$\sum_n (\xi_1\xi_2)^{2n} \times$$

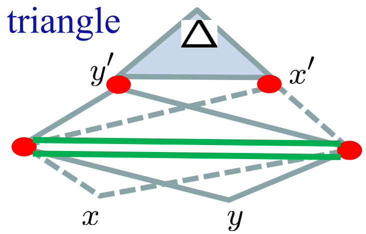



- SO(2,D) spin chain of Length=2.
  Solvable using only conformal symmetry!

We have to diagonalize the graph building operator

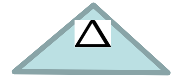
conformal triangle

$x \to \infty$

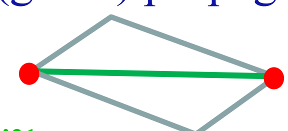



$= h(\Delta)$

$x \to \infty$




Multiplying and dividing by same (green) propagator we get computable "kite" integrals:



Grozin
Derkachev et al'21

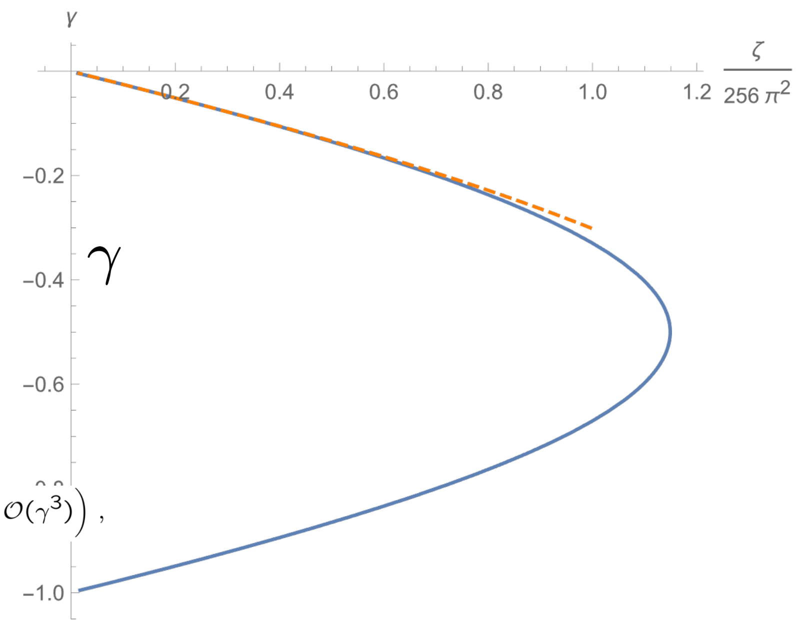
Spectrum of exchange operators in OPE: $h(\Delta) = \frac{1}{\xi^4}$

In the ABJM limit, perturbation theory: $\gamma = \Delta - 2$

$$\frac{1-\gamma}{\xi^4} = -\frac{1}{1024\pi^4\gamma}\left(12\mathfrak{L}_2 + 6\mathfrak{L}_1^2 + \left(12\mathfrak{L}_3 - 2\mathfrak{L}_1^3\right)\gamma + \left(12\mathfrak{L}_4 + \frac{18}{5}\mathfrak{L}_2^2 + \frac{18}{5}\mathfrak{L}_2\mathfrak{L}_1^2 + \frac{7}{5}\mathfrak{L}_1^4\right)\gamma^2 + \mathcal{O}(\gamma^3)\right),$$

$$\mathfrak{L}_j \equiv \mathrm{Li}_j\left(\frac{1}{2}\right)$$

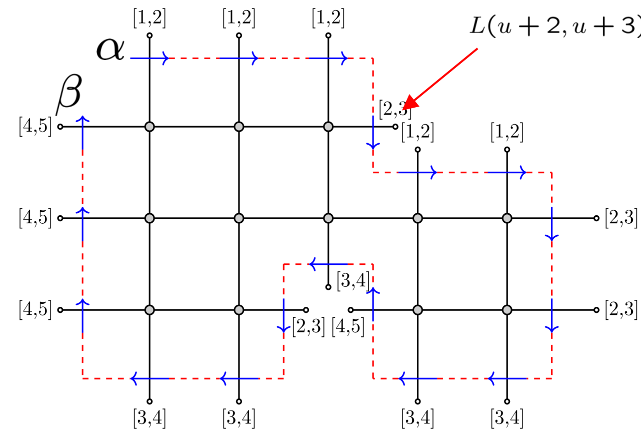
Transcendentality!



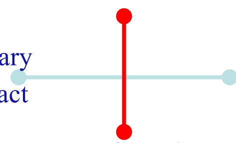
# Yangian symmetry for planar correlators

Chicherin, V.K., Loebbert, Muller, Zhong '17
V.K., Levkovich-Maslyuk, Mishnyakov, '23

- Single-trace correlator in bi-scalar Fishnet CFT is given by a single planar Feynman diagram –disc cut out of regular square lattice:




- "Lasso" operator: product of Lax matrices




$$L(u_+,u_-)=\begin{pmatrix} u_+-\mathbf{px} & \mathbf{p} \\ \mathbf{x}(u_+-u_-)-\mathbf{xpx} & \mathbf{xp}+u_- \end{pmatrix}$$

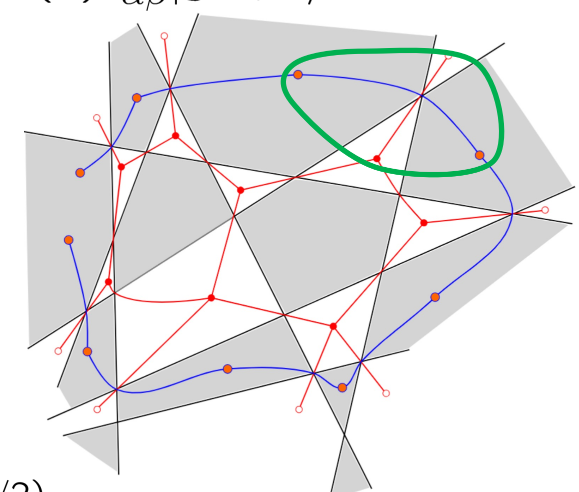
$$\mathbf{x}=-i\bar{\sigma}^\mu x_\mu\ ,\qquad \mathbf{p}=-\frac{i}{2}\sigma^\mu\partial_{x_\mu}$$

$$u_+=u+\frac{\Delta-D}{2}\ ,\quad u_-=u-\frac{\Delta}{2}$$

- Graph is an eigenfunction of Lasso $(L_1L_2\dots L_n)_{\alpha\beta}\,|\mathrm{graph}\rangle=\lambda(u)\delta_{\alpha\beta}|\mathrm{graph}\rangle$

- Generalization to arbitrary loom

V.K., Levkovich-Maslyuk, Mishnyakov, '23



$$(L_n[\delta_n^+,\delta_n^-]\dots L_2[\delta_2^+,\delta_2^-]L_1[\delta_1^+,\delta_1^-])_{\alpha\beta}\,|G\rangle=\delta_{\alpha\beta}\lambda(u)\,|G\rangle$$

- *1/u*$^{n-2}$ term gives Yangian diff. eq.: $\widehat{P}^\mu\,|\mathrm{graph}\rangle=0$

Level-1 momentum operator:

$$\widehat{P}^\mu=-\frac{i}{2}\sum_{j<k}[(L_j^{\mu\nu}+g^{\mu\nu}D_j)P_{k,\nu}-(j\leftrightarrow k)]+\sum_j v_jP_j^\mu$$

evaluation parameters

$$v_k=\frac{1}{2}\sum_{j\neq k}(\delta_j^++\delta_j^-+D/2)$$

- A potentially powerful tool for computation of Feynman graphs

Loebbert, Miczajka, Mueller, Muenkler, '20
Corcoran, Loebbert, Miczajka, Staudacher,'20
Duhr, Klemm, Loebbert, Nega, Porkert,'22

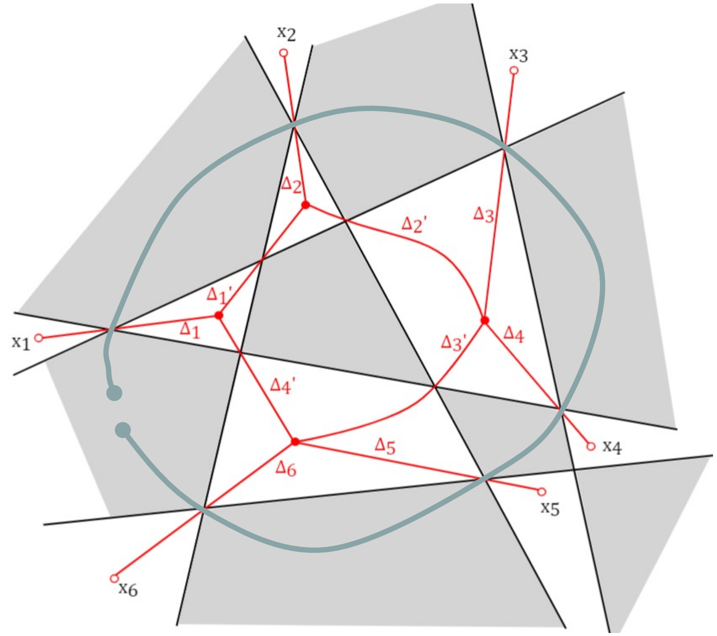
Happy Birthday Fedya!

Joyeux Anniversaire Fedya!

С Юбилеем, Федя!

V.K., Levkovich-Maslyuk, Mishnyakov '23

# Example: 4-loop graph with 6 legs




- We impose: conformality of vertices:

$$\sum_{j\in \text{vertex}_k} \Delta_j^{(k)} = D$$

- Loom (integrability) condition:

$$\Delta_6 + \Delta_2 + \Delta_5 = D$$

- We can express all 10 dimensions in terms of 5 parameters, say

$$\Delta_1, \Delta_2, \Delta_3, \Delta_5, \Delta_1'$$

- Lasso for that graph:

$$L_6[\Delta_{(11')} + D/2, \Delta_{(121'5)}]L_5[\Delta_{(121'5)} - D/2, \Delta_{(121')}]L_4[D, \Delta_{(13)} + D/2] \times L_3[\Delta_{(13)}, \Delta_1 + D/2]L_2[\Delta_{(121')} - D/2, \Delta_{(11')}]L_1[\Delta_1, D/2]$$

where: $\Delta_{(a_1 a_2 \ldots a_p)} = \Delta_{a_1} + \Delta_{a_2} + \cdots + \Delta_{a_p}$

- Evaluation parameters for that graph:

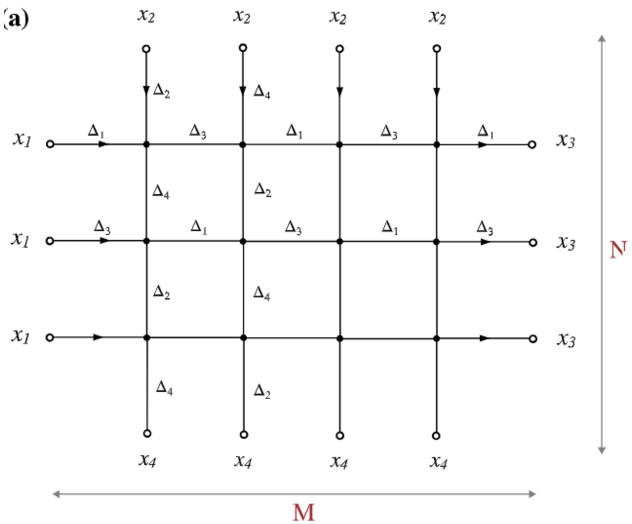
$$v_k = \Big\{0, -\Delta_1' - \frac{\Delta_1}{2} - \frac{\Delta_2}{2} + D/2, -\frac{\Delta_1}{2} - \frac{\Delta_3}{2}, -\frac{\Delta_3}{2} - D/2, -\Delta_1' - \frac{\Delta_1}{2} - \Delta_2 - \frac{\Delta_5}{2} + D/2, -\Delta_1' - \frac{\Delta_1}{2} - \frac{\Delta_2}{2} - \frac{\Delta_5}{2}\Big\}$$

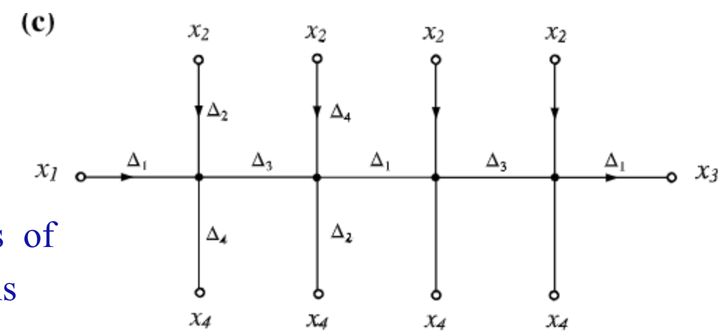
Alfimov, V.K., Ferrando, Olivucci, to appear

# Diamond and Basso-Dixon type graphs for Checkerboard

## 4-point function given by rectangular fishnet graph

$$I_{2n+1,2m} = \langle \mathrm{Tr}((Z_1 Z_3)^n Z_1)(x_1)(Z_2 Z_4)^m(x_2)((\bar{Z}_1\bar{Z}_3)\bar{Z}_1)^n(x_3)(\bar{Z}_2\bar{Z}_4)^m(x_4)\rangle$$



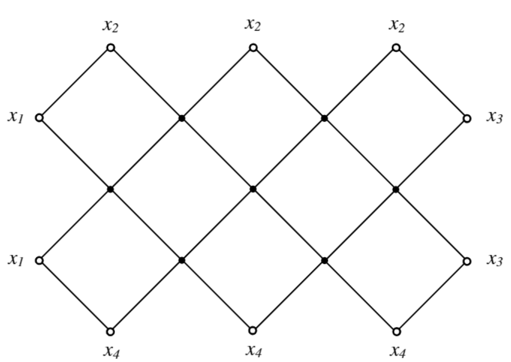




Explicitly computed in terms of a determinant of ladder graphs using the methods of

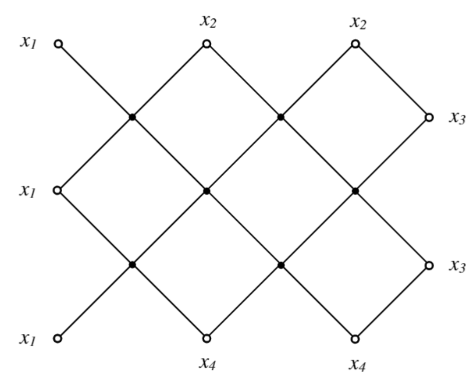
Derkachev, Korchemsky, Manashev 94’
Derkachev, V.K., Olivucci 18’
Derkachev, Olivucci ‘19
Derkachev, Ferrando, Olivucci ‘20

## 4-point functions given by “diamond” fishnet graphs (4 types)

$$G^{(I)}_{m,n} = \langle \mathrm{Tr}(Z_1\bar{Z}_4)^m(x_1)(Z_4 Z_3)^n(x_2)(\bar{Z}_3 Z_2)^m(x_3)(\bar{Z}_2\bar{Z}_1)^n(x_4)\rangle$$




For generic weights,
both are zero for m>n
and tree-like for m=n
Type II is non-trivial for m<n




$$G^{(II)}_{m,n} = \langle \mathrm{Tr}\left[(Z_2\bar{Z}_3)^m(x_1)(Z_4 Z_3)^n(x_2)(\bar{Z}_3 Z_2)^m(x_3)(\bar{Z}_1\bar{Z}_2)^n(x_4)\right]\rangle$$

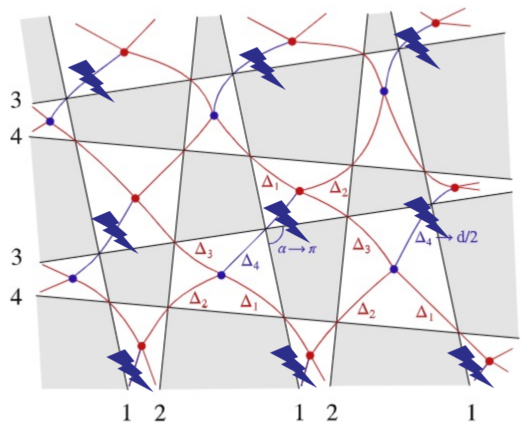
# ABJM and BFKL Fishnet CFTs from FCFT$^{(4)}$

- ABJM FCFT$^{(3)}$ emerges in 3D, for $w_1 = w_2 = w_3 = 1, \quad w_4 = 0$

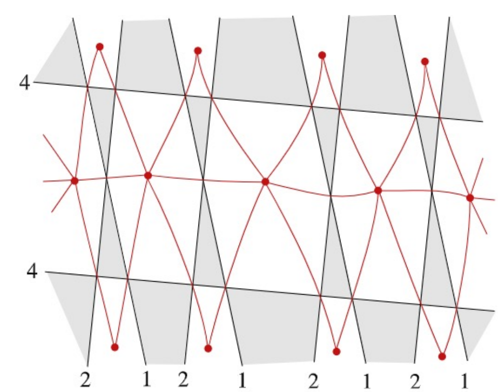
$$\mathcal{L}_D^{(CD)}|_{w_4=0} = N_c \mathrm{tr}\left[-\sum_{j=1}^{3} \bar{X}_j \Box^{w_j} X_j - \bar{X}_4 X_4 + \xi_1^2 \bar{X}_1 \bar{X}_2 X_3 X_4 + \xi_2^2 X_1 X_2 \bar{X}_3 \bar{X}_4\right]$$

$$\to N_c \mathrm{tr}\left[\sum_{j=1}^{3} \partial^\mu \bar{X}_j \partial_\mu X_j + (\xi_1 \xi_2)^2 \bar{X}_1 \bar{X}_2 X_3 X_1 X_2 \bar{X}_3\right]$$

Caetano, Gurdogan, V.K. '16




triangular lattice




- Another interesting case: $D = 2, \quad w_1 = u + 2, \ w_2 = -u, \ w_3 = u, \ w_4 = -u$

  At $u \to 0$ the transfer matrix becomes Lipatov's Hamiltonian for reggeized gluons in QCD

Chicherin, Derkachov, Isaev '12
Alfimov, V.K., Ferrando, Olivucci, to appear