Spectral properties of planar CFT's in D>2: from N=4 SYM to Fishnet CFT

Vladimir Kazakov
A bit of history…

- I met Fedya many years ago in Russia, but we started our scientific discussions (and became friends) only in Paris, in the 90s.

- I learned a lot from him on quantum and classical integrability (in which I was a beginner at the time…)

- Somewhen around 2003, he explained me the finite gap method on the example of his work on KdV field theory. It was crucial for my contribution to one of the decisive works towards complete solution of spectral problem of planar $\mathcal{N}=4$ SYM.

Plan of the talk

- Quantum Spectral Curve (QSC) for planar spectrum of anomalous dimensions in $\mathcal{N}=4$ SYM.

- Fishnet CFT as a weak coupling of strongly $\gamma$-deformed $\mathcal{N}=4$ SYM.

- Generalized “Loom” Fishnet CFT: construction, examples, Yangian symmetry.
Integrability of planar $\mathcal{N}=4$ SYM
\[ N=4 \text{ SYM: planar integrability from AdS/CFT duality} \]

\[ S_{YM} = \frac{1}{\lambda} \int d^4x \, \text{Tr} \left( F^2 + (D\Phi)^2 + [\Phi, \Phi]^2 \right) + \text{fermions} \]

dual to

\[ S \text{ } \boxtimes \text{ } \text{AdS}_5 \]

Super-conformal theory: PSU(2,2|4) symmetry
\[ \beta\text{-function}=0, \text{ no massive particles} \]

Operators

\[ \mathcal{O}(x) = \text{Tr} \left[ \mathcal{D}\mathcal{D}\psi\psi\Phi\Phi\mathcal{D}\psi \ldots \right] (x) \]

Anomalous dimensions

\[ \mathcal{O}(\xi x) \rightarrow \xi^{\Delta(\lambda)} \mathcal{O}(x) \]

Operator product expansion, structure constants, correlators, amplitudes…

1- and 2-loop integrability

Classical integrability

S-matrix, asymptotic Bethe ansatz

\[ \text{Y-system} \quad \text{Thermodynamical Bethe ansatz (TBA)} \quad \text{Quantum spectral curve (QSC)} \]

Gromov, V.K., Vieira '09

Bena, Roiban, Polchinski '02
Metsaev-Tseytlin '02
Minahan, Zarembo '03
Beisert, Kristjansen, Staudacher '03
V.K., Marshall, Minahan, Zarembo '04
Beisert, V.K., Sakai, Zarembo '05

Beisert, Staudacher '04
Beisert '05'
Janik '05'
Beisert, Eden, Staudacher '06

Gromov, V.K., Kozak, Vieira '09
Arutyunov, Frolov '09
Cavaglia, Fioravanti, Tateo '09

Beisert, V.K., Leurent, Volin '13, '14
V.K., Leurent, Volin '15
Quantum Spectral Curve of $\mathcal{N}=4$ SYM: algebraic structure

- QSC eqs. close on finite set of Baxter functions of spectral parameter $Q_I(u)$
- $\text{gl}(n)$: each $Q$ placed at an edge of Hasse diagram - n-hypercube

for $\mathcal{N}=4$ SYM

- QSC eqs. close on finite set of Baxter functions of spectral parameter $Q_I(u)$
- $\text{gl}(n)$: each $Q$ placed at an edge of Hasse diagram - n-hypercube

- Plücker QQ-relations on each face (“determinant flow”):

\[
Q_B(u)Q_D(u) = \begin{vmatrix}
Q_A(u + \frac{i}{2}) & Q_C(u + \frac{i}{2}) \\
Q_A(u - \frac{i}{2}) & Q_C(u - \frac{i}{2})
\end{vmatrix}
\]

- Grassmanian structure!

$\text{SUSY}: \text{gl}(1|2)$

$\text{SUSY}: \text{gl}(8) \rightarrow \text{gl}(4|4)$

- $\text{gl}(N)$ Heisenberg spin chain: $Q_0 = 1, \quad Q_j = \text{Polynomial}(u), \quad Q_{1,2,\ldots,N} \sim u^{\text{Length}}$

- $\text{gl}(M|K)$ Heisenberg spin chain: $Q_{12\ldots M} = 1, \quad Q_{\text{neighbors } 12\ldots M} = \text{Polyn}(u), \quad Q_{M+1,M+2,\ldots,K} \sim u^{\text{Length}}$

Gromov, V.K., Leurent, Volin ‘13,’14
V.K., Leurent, Volin ‘15

Krichever, Lupan, Wiegmann, Zabrodin, ’94
Tsuboi ‘13
Quantum Spectral Curve of AdS$_5$/CFT$_5$: analytic structure

- Special $8 + 8$ Q-functions with nice analyticity on physical sheet

\[ \text{gl}(2,2|4) \]

Hodge dual Q-functions

\[ Q_1 \propto \bar{Q}^2, \quad Q_2 \propto \bar{Q}^1, \quad Q_3 \propto \bar{Q}^4, \quad Q_4 \propto \bar{Q}^3 \]

- Various Q-functions are related by complex conjugation ("gluing" relations)

\[ P_b, \ P^b \sim q^\pm_i u \ (\pm J_1 \pm J_2 \pm J_3)/2 \]

short cut on physical sheet

\[ Q_j, \ Q^j \sim u \ (\pm \Delta \pm S_1 \pm S_2)/2 \]

long cut on physical sheet

- Large $u$ asymptotics fixed by PSU(2,2|4)

Cartan charges $\{\Delta, S_1, S_2 | J_1, J_2, J_3\}$

SU(2,2) SU(4)

- These Riemann-Hilbert conditions fix all physical solutions for Q-system and thus conformal dimensions $\Delta(g)$ with given PSU(2,2|4) charges

Gromov, V.K., Leurent, Volin '13,'14
V.K., Leurent, Volin '15
Dimensions of twist-2,3,… operators  \( \text{Tr}(\Phi \nabla^S \Phi) \)

- Numerics, weak and strong coupling from Quantum Spectral Curve;

Recent results “unlimited” precision (~30–50 digits)

- QSC for ABJM, AdS3/CFT2

\[
\Delta_{\text{Konishi}} \simeq 2\lambda^{1/4} + \frac{2}{\lambda^{1/4}} + \frac{-3\zeta_3 + \frac{1}{2} \zeta_5 + \frac{15}{2} \zeta_3 - \frac{3}{2}}{\lambda^{5/4}} + \ldots
\]

\( \lambda = 16\pi^2 g^2 \)

\[
\gamma_{\text{Konishi}} = \sum_{j=1}^{\infty} g^{2j} \gamma_j
\]

Weak coupling (11 loops)

Gromov, Julius, Sokolova ‘23

\[ g = 0.3 \]

Gromov, V.K., Vieira ‘09
Frolov ‘10
Gromov, Valatka ‘12

Gromov, V.K., Leurent ‘13
Leurent, Serban, Volin ‘12
Volin, Marboe ‘18

\[
\Delta(S,g)
\]

Function of complex conformal spin

Gromov, Levkovich-Maslyuk, Sizov ‘15

Gromov, Valatka, Sizov, Levkovich-Maslyuk
Gromov, Shenderovich, Serban, Volin
Roiban, Tseytlin
Valillo, Mazzucato
Gubser, Klebanov, Polyakov

\[
L = 3, S = 2
\]

\[
L = 2, S = 2 \text{ (Konishi)}
\]

\[
\text{Strong coupling:}
\]

\[ g = 0.3 \]
Fishnet CFT
\( \gamma \)-twisted N=4 SYM and “fishnet” limit

\[
L = N_c \text{tr} \left( F^2 + D\phi_i D\phi_i + i\psi_j \slashed{D}\psi_j + i\slashed{\lambda} \lambda + g^2 [\phi_j, \phi_k]_q \cdot [\bar{\phi}_j, \bar{\phi}_k]_q + ig \epsilon_{ijk}\psi_k [\phi_i, \psi_j]_q + g \bar{\lambda} [\phi_j, \bar{\psi}_j]_q + \text{conj.} \right)
\]

• \( \gamma \)-twisted N=4 SYM Lagrangian: commutators \( \rightarrow \) q-commutators

\[
[A, B] \rightarrow [A, B]_q = q_{AB} AB - \frac{1}{q_{AB}} BA
\]

where \( q_{A,B} = e^{-\frac{i}{2} \epsilon^{mjk} \gamma_m J_j^A J_k^B} = (q_{B,A})^{-1} \)

\( J_1^A, J_2^A, J_3^A \in SO(6) \) - Cartan charges of R-symmetry

\( \gamma_1, \gamma_2, \gamma_3 \) - twists

• Breaks R-symmetry and all supersymmetry: PSU(2,2|4)\( \rightarrow \) SU(2,2) \( \times \) U(1)\(^3\)

• Double scaling “fishnet” limit: Strong imaginary twist, weak coupling:

\( g \rightarrow 0, \quad \gamma_j \rightarrow i\infty, \quad \xi_j = g e^{-i\gamma_j/2} \) - fixed, \( (j = 1, 2, 3.) \)

• Planar Feynman graphs form a dynamical fishnet: solid lines – bosons, dashed lines - fermions

• Integrable, since deduced from integrable N=4 SYM! But how to see it explicitly?
Special case: bi-scalar Fishnet CFT$_4$

- Retain only one coupling in fishnet limit of N=4 SYM:
  \[ \xi_1 = \xi_2 = 0, \quad \xi_3 \equiv \xi \neq 0 \]

- $N = 4$ SYM planar graphs reduce, in the bulk, to (very few!) fishnet graphs.

- No mass or vertex renormalization! Coupling $\xi$ does not run!

- But there are double-trace counterterms with $\xi$ dependent couplings:
  \[ \text{tr} \left( \bar{\phi}_1 \phi_2 \right) \text{tr} \left( \bar{\phi}_2 \phi_1 \right), \quad \text{tr} \left( \phi_1 \bar{\phi}_2 \right) \text{tr} \left( \bar{\phi}_1 \phi_2 \right), \quad \text{tr} \left( \phi_j \bar{\phi}_j \right) \text{tr} \left( \bar{\phi}_j \phi_j \right) \]

- One can study correlators of local operators
  \[ \mathcal{O}(x) = C^{\mu_1 \cdots \mu_n} \text{tr} \left[ \partial_{\mu_1} \cdots \partial_{\mu_n} (\phi_1)^L (\phi_2)^M (\bar{\phi}_1)^K (\bar{\phi}_2)^N \right](x) + \text{permutations} \]
Operators, correlators, graphs…

- Explicit computations of correlators
  “vacuum” operator
  \[
  \text{tr}[\phi(x)]^L
  \]
  Multi-magnon spiral graphs
  Gurdogan, V.K ‘15
  Caetano, Gurdogan, V.K ‘16

- Basso-Dixon 4-point functions through determinant of “ladder” graphs, Sklyanin SoV
  \[
  G_{m,n}(x_1, x_2, x_3, x_4) = x_1
  \]
  Davidichev, Ushuikina
  Basso, Dixon
  Derkachev, V.K., Olivucci
  Derkachev, Ferrando, Olivucci ‘21
  Dercachov, Olivucci ‘21
  Basso, Dixon, Kosover, Krajnenbrink, Zhong ‘21
  Kostov ‘23
  ...

- Amplitudes, Yangian symmetry, Calabi-Yau periods…
  Chicherin, V.K., Mueller, Loebbert, Zheng ‘17
  Corcoran, Loebbert, Miczajka, Muller, Munkler ‘20
  Duhr, Klemm Loebbert, Nega, Porkert’22
  V.K., Levkovich-Maslyuk, Mishnyakov ‘23

- Thermodynamical Bethe Ansatz for Fishnet
  “Fishchain”: AdS dual for Fishnet
  Non-trivial flat vacua in Fishnet CFT
  Eclectic spin chain from Fishnet
  Beyond Fishnet limit of N=4 SYM
  Basso, Zhong ‘19
  Basso, Ferrando, V.K., Zhong ‘19
  Gromov, Sever ‘19
  V.K., Karananas, Shaposhnikov ‘19
  Ipsen, Staudacher, Zippelius ‘18
  Ahn, Staudacher ‘21, ‘22
  Ferrando, Sever ‘23

- Thermodynamics of Fishnet graphs
  Zamolodchikov ‘80
  Staudacher, Kade ‘23

• Operators, correlators, graphs…

\[
\text{tr}[\phi_1(x_1) \phi_1(x_2)]
\]

\[
\text{tr}[\phi_1^\dagger(x_3) \phi_1^\dagger(x_4)]
\]

OPE, 4-point functions, stampedes…

Grabner, Gromov, V.K., Korchemsky ‘17
V.K., Olivucci 2018
Gromov, V.K., Korchemsky ’18
V.K., Olivucci, Preti, ’19
Pitelli, Preti ’19
Gromov, Sever ‘20
Olivucci, Vieira ‘22
Olivucci ‘23
Dimension of $\text{tr}(\phi_1)^3$ and periods of wheel graphs from QSC

Interesting relation to Galois coaction on Feynman periods

Based on Quantum Spectral curve of $\mathcal{N}=4$ SYM

Baxter eq.:

$$
\left( \frac{(\Delta - 1)(\Delta - 3)}{4u^2} - \frac{i\xi^3}{u^3} - 2 \right) q(u) + q(u + i) + q(u - i) = 0
$$

Asymptotics:

$$
q_1(u, \xi) \sim u^{\Delta/2 - 1/2}(1 + \frac{\alpha_1}{u} + \frac{\alpha_2}{u^2} + \cdots)
$$

$$
q_2(u, \xi) \sim u^{-\Delta/2 + 3/2}(1 + \frac{\beta_1}{u} + \frac{\beta_2}{u^2} + \cdots)
$$

Quantization condition

$$
q_1(0, \xi) q_2(0, -\xi) + q_2(0, -\xi) q_1(0, \xi) = 0
$$
High precision numerics for spectrum and “PT” symmetry

The two dimensions are real for $\xi < \xi_c$, but they turn to complex conjugates for $\xi > \xi_c$

The reason for this reality of spectrum: “PT” symmetry of Fishnet CFT

“PT”-transformation leaves the action invariant (but not operators!):

Conformal dimension gets complex conjugate (non-unitary theory!):

$$\left[ \langle \bar{O}(x)O(0) \rangle \right]^{PT} = \langle \bar{O}^{PT}(x)O^{PT}(0) \rangle = |x|^{-2\Delta^*}$$

The spectrum consists of real dimensions and/or of complex conjugate pairs!

Similar to energy spectrum of non-unitary PT-invariant quantum mechanics

$$\mathcal{H} = \hat{p}^2/2 + x^2 (ix)^{\epsilon}$$
Loom for fishnet CFTs from Baxter lattices

- Baxter lattice for general Fishnet CFT: \( M \) intersecting lines with \( M \) slopes

\[ G_B = \int \prod_{m \in L_I} d^D x_m \prod_{\langle j,k \rangle \in L_I} G_D(x_j, x_k, \alpha_{jk}) \]

Integrability of such “loom” graphs based on star-triangle relation

\[ G_D(x_j, x_k, \alpha_{jk}) = |x_j - x_k| \frac{D}{\pi} (\alpha_{jk} - \pi) \]

All Feynman graphs of the loom are connected by star-triangle relations!

We will construct Fishnet CFT\(^{(M)}\) with all such Feynman graphs
Generalized Fishnet CFT: Kinetic Terms

To accommodate all these graphs within a Fishnet CFT\(^{(M)}\), we need \(M(M-1)\) scalar fields (two for each crossing)

Kinetic terms are defined by dimensions of fields:

\[
\mathcal{L}_{\text{kin}} = \frac{N_c}{2} \text{tr} \left( - \sum_{j=1}^{M(M-1)} \bar{\phi}_i \left( \square^{D/2-\Delta \phi_i} \right) \phi_i \right)
\]

Example of \(M=3\):

6 fields with dimensions

\[
\begin{align*}
\Delta_X &= \frac{D}{2} \\
\Delta_Y &= \frac{D}{2} \\
\Delta_Z &= \frac{D}{2} - \Delta_X - \Delta_Y \\
\Delta_u &= \frac{D}{2} - \Delta_X \\
\Delta_v &= \frac{D}{2} - \Delta_Y \\
\Delta_w &= \Delta_X + \Delta_Y
\end{align*}
\]
Generalized Fishnet CFT: Interactions

Construct the vertices: start from the largest one and consecutively replace pairs of fields by the “dual” fields (according to star-triangle):

For example, general FCFT\(^{(3)}\) has 18 vertices:

1 sextic
\[ \text{tr} \left( X Y Z \bar{X} \bar{Y} \bar{Z} \right) \]

6 quintic
\[ \text{tr} \left( w Z \bar{X} \bar{Y} \bar{Z} \right) \quad \text{tr} \left( X u \bar{X} \bar{Y} \bar{Z} \right) \quad \text{tr} \left( X Y v \bar{Y} \bar{Z} \right) \]
\[ \text{tr} \left( X Y Z \bar{w} \bar{Z} \right) \quad \text{tr} \left( X Y Z \bar{X} \bar{u} \right) \quad \text{tr} \left( Y Z \bar{X} \bar{Y} \bar{v} \right) \]

9 quartic
\[ \text{tr} \left( w v \bar{Y} \bar{Z} \right) \quad \text{tr} \left( w Z \bar{X} \bar{u} \right) \quad \text{tr} \left( X Y v \bar{u} \right) \]
\[ \text{tr} \left( u \bar{X} \bar{Y} \bar{v} \right) \quad \text{tr} \left( X Y v \bar{u} \right) \quad \text{tr} \left( Y Z \bar{w} \bar{v} \right) \]
\[ \text{tr} \left( Y v \bar{Y} \bar{v} \right) \quad \text{tr} \left( w Z \bar{w} \bar{Z} \right) \quad \text{tr} \left( u Z \bar{X} \bar{u} \bar{Z} \right) \]

2 cubic
\[ \text{tr} \left( v \bar{w} \bar{u} \right) \quad \text{tr} \left( w v \bar{u} \right) \]

There are also double-trace terms…

These “Loom” Fishnet CFTs have integrable planar graphs at any D!
Checkerboard FCFT\(^{(4)}\)

A Loom FCFT with M=4 slopes but only two interaction terms turned on:

\[
\mathcal{L}^{(CD)}_D |_{w_4=0} = N_c \text{tr} \left[ - \sum_{j=1}^{4} \bar{X}_j \Box w_j X_j + \xi_1^2 \bar{X}_1 \bar{X}_2 X_3 X_4 + \xi_2^2 X_1 X_2 \bar{X}_3 \bar{X}_4 \right], \quad \sum_{j=1}^{4} w_j = D
\]

R-matrix in principals series irrep of conformal group

- Local action in 4D \( w_1 = w_2 = w_3 = w_4 = 1 \)
- ABJM FCFT\(^{(3)}\) emerges in 3D, for \( w_1 = w_2 = w_3 = 1, \ w_4 = 0 \)
  \[
  \mathcal{L}^{(ABJM FCFT)}_3 = N_c \text{tr} \left[ \sum_{j=1}^{3} \partial^\mu \bar{X}_j \partial_\mu X_j + (\xi_1 \xi_2)^2 \bar{X}_1 \bar{X}_2 X_3 X_1 X_2 \bar{X}_3 \right]
  \]
- At \( u \to 0 \) for \( D = 2 \), \( w_1 = u + 2, \ w_2 = -u, \ w_3 = u, \ w_4 = -u \)
  transfer matrix becomes Lipatov’s Hamiltonian for reggeized gluons in QCD

\[ a = |x_1 - x_2|^{-2a}\]

Satisfies Yang-Baxter relation:

Derkachov, Korchemsky, Manashov ‘01
Chicherin, Derkachev, Isaev’12
Caetano, Gurdogan, V.K., ‘16
4-point correlator for Checkerboard FCFT\(^{(4)}\)

\[
\langle \text{tr} [X_1 X_2(x) \ X_1 X_2(y)] \ \text{tr} [\bar{X}_3 \bar{X}_4(x') \ \bar{X}_3 \bar{X}_4(y')] \rangle_c
\]

- Given by ladder graphs (each square is R-matrix)
- SO(2,D) spin chain of Length=2.
  Solvable using only conformal symmetry!

We have to diagonalize the graph building operator

We get computable “kite” integrals:

\[
\sum_n (\xi_1 \xi_2)^{2n} \times \n \int \Delta = h(\Delta)
\]

Multiplying and dividing by same (green) propagator

Spectrum of exchange operators in OPE: \( h(\Delta) = \frac{1}{\xi^4} \)

In the ABJM limit, perturbation theory: \( \gamma = \Delta - 2 \)

\[
\frac{1 - \gamma}{\xi^4} = -\frac{1}{1024\pi^4\gamma} \left( 12\xi_2 + 6\xi_3^2 + \left( 12\xi_3 - 2\xi_4^2 \right) \gamma + \left( 12\xi_4 + \frac{18}{5}\xi_2^2 + \frac{18}{5}\xi_2\xi_1 + \frac{7}{5}\xi_1^2 \right) \gamma^2 + O(\gamma^3) \right)
\]

\( \xi_j \equiv \text{Li}_j \left( \frac{1}{2} \right) \)

Transcendentality!
Yangian symmetry for planar correlators

- Single-trace correlator in bi-scalar Fishnet CFT is given by a single planar Feynman diagram – disc cut out of regular square lattice:

  \[ L(u_+, u_-) = \left( \begin{array}{c} u_+ - px \\ x(u_+ - u_-) - xp \\ xp + u_- \end{array} \right) \]

  \[ x = -i \bar{\sigma}^\mu x_\mu, \quad p = -\frac{i}{2} \sigma^\mu \partial x_\mu \]

  \[ u_+ = u + \frac{\Delta - D}{2}, \quad u_- = u - \frac{\Delta}{2} \]

- “Lasso” operator: product of Lax matrices

  \[ (L_1 L_2 \ldots L_n)_{\alpha \beta} |\text{graph}\rangle = \lambda(u) \delta_{\alpha \beta} |\text{graph}\rangle \]

- Graph is an eigenfunction of Lasso

- Generalization to arbitrary loom

  \[ (L_n [\delta_n^+, \delta_n^-] \ldots L_2 [\delta_2^+, \delta_2^-] L_1 [\delta_1^+, \delta_1^-])_{\alpha \beta} |G\rangle = \delta_{\alpha \beta} \lambda(u) |G\rangle \]

- \(1/\mu^{n-2}\) term gives Yangian diff. eq.: \(\hat{P}^\mu |\text{graph}\rangle = 0\)

  Level-1 momentum operator:

  \[ \hat{P}^\mu = -\frac{i}{2} \sum_{j<k} [(L_j^{\mu \nu} + g^{\mu \nu} D_j) P_{k,\nu} - (j \leftrightarrow k)] + \sum_j v_j P_j^\mu \]

  Evaluation parameters

  \[ v_k = \frac{1}{2} \sum_{j \neq k} (\delta_j^+ + \delta_j^- + D/2) \]

- A potentially powerful tool for computation of Feynman graphs

Chicherin, V.K., Loebbert, Muller, Zhong '17
V.K., Levkovich-Maslyuk, Mishnyakov, '23

Loebbert, Miczajka, Mueller, Muenkler, '20
Corcoran, Loebbert, Miczajka, Staudacher, '20
Duhr, Klemm, Loebbert, Nega, Porkert, '22
Happy Birthday Fedya!
Joyeux Anniversaire Fedya!
С Юбилеем, Федя!
Example: 4-loop graph with 6 legs

- We impose: conformality of vertices:

\[ \sum_{j \in \text{vertex}_k} \Delta^{(k)}_j = D \]

- Loom (integrability) condition:

\[ \Delta_6 + \Delta_2 + \Delta_5 = D \]

- We can express all 10 dimensions in terms of 5 parameters, say \( \Delta_1, \Delta_2, \Delta_3, \Delta_5, \Delta'_1 \)

- Lasso for that graph:

\[ L_6[\Delta_{(11')} + D/2, \Delta_{(121'5)}] L_5[\Delta_{(121'5)} - D/2, \Delta_{(121')} ] L_4[D, \Delta_{(13)} + D/2] \times L_3[\Delta_{(13)} , \Delta_1 + D/2] L_2[\Delta_{(121')} - D/2, \Delta_{(11')} ] L_1[\Delta_1, D/2] \]

where: \( \Delta_{(a_1a_2...a_p)} = \Delta a_1 + \Delta a_2 + \cdots + \Delta a_p \)

- Evaluation parameters for that graph:

\[
v_k = \left\{ 0, -\Delta'_1 - \frac{\Delta_1}{2} - \frac{\Delta_2}{2} + D/2, -\frac{\Delta_1}{2} - \frac{\Delta_3}{2}, -\frac{\Delta_3}{2} - D/2, -\Delta'_1 - \frac{\Delta_1}{2} - \Delta_2 - \frac{\Delta_5}{2} + D/2, -\Delta'_1 - \frac{\Delta_1}{2} - \frac{\Delta_2}{2} - \frac{\Delta_5}{2} \right\}
\]
Diamond and Basso-Dixon type graphs for Checkerboard

4-point function given by rectangular fishnet graph

\[ I_{2n+1,2m} = \langle \text{Tr}((Z_1 Z_3)^n Z_1)(x_1)(Z_2 Z_4)^m(x_2)((\bar{Z}_1 \bar{Z}_3) \bar{Z}_1)^n(x_3)(\bar{Z}_2 \bar{Z}_4)^m(x_4) \rangle \]

Explicitly computed in terms of a determinant of ladder graphs using the methods of

Derkachev, Korchemsky, Manashev 94'
Derkachev, V.K., Olivucci 18'
Derkachev, Olivucci ’19
Derkachev, Ferrando, Olivucci ’20

4-point functions given by “diamond” fishnet graphs (4 types)

\[ G^{(I)}_{m,n} = \langle \text{Tr}((Z_1 \bar{Z}_4)^m(x_1)(Z_4 Z_3)^n(x_2)(\bar{Z}_3 Z_2)^m(x_3)(\bar{Z}_2 \bar{Z}_1)^n(x_4) \rangle \]

For generic weights, both are zero for m>n and tree-like for m=n
Type II is non-trivial for m<n

\[ G^{(II)}_{m,n} = \langle \text{Tr} [(Z_2 \bar{Z}_3)^m(x_1)(Z_4 Z_3)^n(x_2)(\bar{Z}_3 Z_2)^m(x_3)(\bar{Z}_1 \bar{Z}_2)^n(x_4)] \rangle \]
ABJM and BFKL Fishnet CFTs from FCFT\(^{(4)}\)

- **ABJM FCFT\(^{(3)}\)** emerges in 3D, for \(w_1 = w_2 = w_3 = 1, \quad w_4 = 0\)

\[
\mathcal{L}^{(CD)}_{D=0} = N_c \text{tr} \left[ -\sum_{j=1}^{3} \bar{X}_j \Box^{w_j} X_j - \bar{X}_4 X_4 + \xi_1^2 \bar{X}_1 \bar{X}_2 X_3 X_4 + \xi_2^2 X_1 X_2 \bar{X}_3 \bar{X}_4 \right] \\
\rightarrow N_c \text{tr} \left[ \sum_{j=1}^{3} \partial^\mu \bar{X}_j \partial_\mu X_j + (\xi_1 \xi_2)^2 \bar{X}_1 \bar{X}_2 X_3 X_1 X_2 \bar{X}_3 \right]
\]

Caetano, Gurdogan, V.K. ‘16

- **Another interesting case:** \(D = 2, \quad w_1 = u + 2, \quad w_2 = -u, \quad w_3 = u, \quad w_4 = -u\)

At \(u \rightarrow 0\) the transfer matrix becomes Lipatov’s Hamiltonian for reggeized gluons in QCD

Chicherin, Derkachov, Isaev ‘12
Alfimov, V.K., Ferrando, Olivucci, to appear