How to control the breaking of integrability

Giuseppe Mussardo
SISSA & INFN
Trieste
Topics of the seminar

• The problem of the spectrum

• The art of integrable models
  (i) Some renowned examples
  (ii) Experimental signatures

• Breaking integrability
  (i) Truncated Hamiltonian approaches
  (ii) Semi-classical methods
  (iii) Form Factor Perturbation Theory

• Confinement
  (i) Fragility of the kinks
  (ii) The touch of supersymmetry
On a general footing, determining the spectrum of a QFT is quite a difficult problem...

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_c \bar{q}_c \left[ i\gamma^\mu (\partial_\mu - igA_\mu) - m_c \right] q_c \]
Few methods are usually available to study the dynamical spectrum (i.e. bound states) of a QFT
Bethe-Salpeter Equation (1951)
$N_t = 3$
$\beta = 5.4$
$K_\eta = 0.163 \quad \mu_\eta = 3.0675\ldots$
In two dimensions the exact spectrum can be extracted by looking at the poles of the exact S-matrix, when the system is integrable.
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(Sine-Gordon model, Ising model in a magnetic field, Yang-Lee model, Tricritical Ising Model, Potts model, etc.)
A remarkable example: 2d Ising model in a magnetic field at $T=T_c$ ...
...and its underline $E_8$ structure
INTEGRALS OF MOTION AND S-MATRIX OF THE (SCALED) $T = T_c$ ISING MODEL WITH MAGNETIC FIELD

A. B. ZAMOLDOCHIKOV$^a$

$^a$ Northwestern University, Evanston, Illinois, USA.

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The spin–spin correlation function in the two-dimensional Ising model in a magnetic field at $T = T_c$

G. Dellino$^b$, G. Mussardo$^b$

$^b$ International School for Advanced Studies, Trieste, Italy.

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Abstract

The spin–spin correlation function in the two-dimensional Ising model is evaluated at the critical temperature $T = T_c$. The matrix elements of the magnetization operator are estimated by solving the spin-staggered equations of the two-dimensional Ising model, taking into account the singularities of the model. The exact solution is conjectured to be of the form

$$\phi(x) = \sum_{n} a_n x^n$$

where $a_n$ are constants. The two-dimensional Ising model is used to study the quantum fluctuations of the model. The results are compared with the exact solution.

1. Introduction

Over the past few years, considerable progress has been made in the use of conformal invariance methods and scattering theory for the study of the critical point and the nearby scaling region of two-dimensional models. The results are well described by the fixed point equations. The critical point is described by the renormalization group equations. The results are compared with the exact solution.

$$\phi(x) = \sum_{n} a_n x^n$$

where $a_n$ are constants. The two-dimensional Ising model is used to study the quantum fluctuations of the model. The results are compared with the exact solution.
whether this system can host a complete picture of SrCo magnetic (AFM) materials, e.g., BaCo materials-based studies on this fascinating phenomenon dynamics and the excitations in quantum magnets. Experimental test bed for an exploring exotic feature of the direction. We use NMR to accurately locate the 1D QCP measurements on BCVO with a field along the [010] axis. We present the full mental realization of the modes in the studied energy window. Furthermore, our study also captures all the multiparticle of the essential integrable part of the model, providing an Eq. highly consistent with both the numerical calculations with.

Along this line, excitations up to the fifth power of the eight massive states perturbed away from the 1D QCP. [see Fig. 1(b)], with a 3D QCP at $E_s \sim T$ and a "...". It is desirable to explore, the quasi-1D Heisenberg-Ising antiferro-magnetic (AFM) materials, e.g., BaCo materials-based studies on this fascinating phenomenon dynamics and the excitations in quantum magnets. Experimental test bed for an exploring exotic feature of the direction. We use NMR to accurately locate the 1D QCP measurements on BCVO with a field along the [010] axis. We present the full mental realization of the modes in the studied energy window. Furthermore, our study also captures all the multiparticle of the essential integrable part of the model, providing an Eq. highly consistent with both the numerical calculations with.

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Observation of $Z_{\nu}$ particles in an Au-Au chain antiferromagnet


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We report the observation of $Z_{\nu}$ particles in an Au-Au chain antiferromagnet. The $Z_{\nu}$ particles are identified by the measurement of their magnetic moment and spin angular momentum using neutron scattering experiments. The results are consistent with the predictions of $Z_{\nu}$ particles in the framework of the microscopic model for the Au-Au chain antiferromagnet. These results provide a new possibility for the realization of quantum computation based on quantum spin chains.
The curves in Fig. 3 represent the results with and without the zone-folding effect. DSF spectra for individual scattering channels are shown: (d) single-particle, (e) two-particle, (f) three-particle, and (g) four-particle contributions. The peak with mass determined from the drop in the phase diagram follows an analytical form of magnetic phase at high-temperature. The red vertical lines at eight peaks correspond to the eight single-mass spin-wave excitations. The peak with mass \( m \) at each field is then performed INS to measure the dynamical structure factor (DSF) of BCVO under the transverse magnetic field. Below 15 K, the competition between the 1D quantum criticality and the 3D ordered background results in interesting excitation. We demonstrate in the inset of Fig. 3(a) that the linearly vanishing gap and scaling exponent 0.75 near 5 T follow the power-law form at \( \mu_0 H = 4.7 \) T, consistent with results from other works. From 6 to 12 K, the scaling exponent deviates from it at other fields, hence, a 1D QCP is established in BCVO at the vicinity of the 1D QCP. Since the 1D QCP hides in the 3D ordering dome, the TFIC universality in the relaxation rate is better resolved in BCVO than in other systems. From the height of the peak at 0.4 K, the blue diamonds with error bars correspond to experimental data and black lines are the fit to the Gaussian.
Integrable vs NonIntegrable QFT

Elastic process

Production processes
Analytic structure of the S-matrix
• Can we determine the number of stable scalar particles around each vacuum?

• What is the value of their mass?

• What is the decay rate of the resonances?

• Can we compute the elastic part of the S-matrix of the kinks below threshold?

• How much stable are the kinks?!
Caution

Ordinary Perturbation Theory may give wrong answer

• Sine-Gordon model

\[ V(\phi) = \frac{m^2}{\beta^2} (1 - \cos \beta \phi) \approx \frac{m^2}{2} \phi^2 + \cdots \]

It seems that there is always a scalar excitation on each vacuum

But, from the exact solution of the model, there are actually none if

\[ \beta^2 \geq 4\pi \]
Although not exactly solvable, a big deal of information can be also extracted when the systems are non-integrable.
• Truncated Conformal Space Approach
  (Quite an efficient numerical method, different from Montecarlo)

• Semi-classical method
  (It works particularly well in presence of kink excitations)

• Form-Factor Perturbation Theory
  (Well suited for studying the breaking of integrable models and the evolution of their spectrum)
Truncated Conformal Space Approach

Matrix elements on the conformal states

\[ H = H_0 + \lambda \int_0^R dx \phi(x) = H_{CFT} + V \]

\[ < n | H_{CFT} | m > = \frac{2\pi}{R} (\Delta_n + \tilde{\Delta}_n - \frac{c}{12}) \delta_{n,m} \]

\[ < n | V | m > = \lambda \left(\frac{2\pi}{R}\right)^{2\Delta_n - 1} C'_{nm} \]
Behavior of the eigenvalues

\[ E_i \sim \begin{cases} \frac{2\pi}{R} \left( \Delta_i + \overline{\Delta}_i - \frac{c}{12} \right), & R \ll \xi \\ \frac{c_0}{\xi^2} R + \sum_i m_i, & R \gg \xi \end{cases} \]
Role of the boundary conditions

• Kink state exist only with anti-periodic or twisted b.c.
Role of the boundary conditions

- Only kink-antikink states are present with periodic b.c.
Integrable – non integrable signatures

Integrable case

Non-integrable case
Integrable vs non-integrable
Such aspects have been analysed in deformed CFT (Brandino, Konik, GM).

\[ H = H_{\text{CFT}} + (T - T_c) \int \sigma \left( x \right) \, d\xi \]
\[ \int_{\psi=0}^{2\pi} \int_{\phi=0}^{\varphi} P_{\alpha} d\psi = j \hbar \int_{\psi=0}^{\varphi} P_{\alpha} d\varphi \]

\[ E_{j,\kappa} = -\frac{2n^2m_e^2Z^2 \frac{1}{(j+\kappa)^2}}{\hbar^2} \]

\[ j + \kappa = n \]

\[ E_n = -\frac{2n^2m_e^2Z^2}{\hbar^2} \frac{1}{\mu^2} \]

\[ \hbar \nu_{m,n} = E_{n'} - E_n \]
Semi-classical methods

\[ L = \frac{1}{2} \left( \partial_{\mu} \Phi \right)^{2} - U(\Phi) \]

Where \( U(\Phi) \) has a set of degenerate vacua, e.g.
Semi-classical methods

\[ L = \frac{1}{2} \left( \partial_{\mu} \Phi \right)^2 - U(\Phi) \]

Where \( U(\Phi) \) has a set of degenerate vacua, e.g.

(i) Sine-Gordon

\[ U(\varphi) = \frac{m^2}{\beta^2} (1 - \cos \beta \varphi) \]

(ii) Broken phase of \( \Phi^4 \) theory

\[ U(\varphi) = \frac{\lambda}{4} \left( \varphi^2 - \frac{\mu^2}{\lambda^2} \right)^2 \]

(iii) Multi-frequency Sine-Gordon

\[ U(\varphi) = \mu \cos \beta \varphi + \lambda \cos \omega \varphi \]
The method works equally well also in presence of fermions

\[ \mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - U(\phi) + i \bar{\psi} \gamma^\mu \partial_\mu \psi - g V(\phi) \bar{\psi} \psi \]

**SUSY:**

\[ U(\phi) = [W'(\phi)]^2 \quad V(\phi) = W''(\phi) \]
Basic excitations: kinks and anti-kinks

\[ | K_{a,a+1}(\theta) \rangle \]
Q-topological sector

\[ Q = \varphi(\infty) - \varphi(-\infty) \]
The classical kink configurations are obtained by integrating

\[
\frac{1}{2} \left( \frac{\partial \Phi_{cl}}{\partial x} \right)^2 = V(\Phi_{cl}) + C
\]
Semi-classical data

\[ M_{cl} = \begin{cases} \frac{8m}{\beta^2} & \text{SG} \\ \frac{2\sqrt{2}}{3} \frac{m^3}{\lambda} & \Phi^4 \end{cases} \]
Mass of the kinks: rule of thumb and BPS

\[ M_{ab} = \int_{\varphi_a}^{\varphi_b} \sqrt{2U(\varphi)} \, d\varphi = Z_{ab} \]
Bosonic Kink Form Factor

(Goldstone-Jackiw, GM)

The semi-classical two-particle matrix element of the fundamental field among kink states is simply the Fourier transform of the classical solution

$$\langle K_{ab}^{\pm}(\theta_1) \mid \varphi(0) \mid K_{ab}^{\pm}(\theta_2) \rangle \equiv F(\theta)$$

$$\theta = \theta_1 - \theta_2$$

$$F(\theta) = \int da \ e^{iM_{cl}^a \theta} \Phi_{cl}(a)$$
Its proof is simple

- Heisenberg equation of motion

\[ (\partial_t^2 - \partial_x^2) \Phi(x, t) = - V'[\Phi(x, t)] \]
Its proof is simple

- Heisenberg equation of motion

\[(\partial_t^2 - \partial_x^2) \langle K(\theta_1)|\Phi(x,t)|K(\theta_2) \rangle = -\langle K(\theta_1)|V'[\Phi(x,t)]|K(\theta_2) \rangle\]

- Extract the \((x,t)\) dependence and use the rapidity difference to define

\[f(a) = \int \frac{d\theta}{2\pi} e^{-iM_{cl}\theta a} F(a)\]

\[F(\theta) \equiv \langle K(\theta_1)|\Phi(0)|K(\theta_2) \rangle\]

\[\theta = \theta_1 - \theta_2\]
• Left hand side

\[ 2 M_{cl}^2 (1 - \cosh \theta) \simeq -(M_{cl} \theta)^2 F(\theta) \]

• Right hand side (semi-classically) is saturated by the lowest intermediate states

• Hence \( f(a) \) satisfies the same differential equation satisfied by the classical kink solution

\[ \frac{d^2}{da^2} f(a) = V'[f(a)] \]
Semi-classical Form Factor of other bosonic operators

\[ F_G^G (\theta) = \int da \, e^{iM\theta a} \, G[\Phi_{cl}(a)] \]

Example

\[ F_{\Phi^2} (\theta) = \int da \, e^{iM\theta a} \, \Phi_{cl}^2 (a) \]
Bound states

The semi-classical expression of the Form Factors permits to easily obtain the bound states of the theory.
Bound states

To this aim, consider the poles of the crossed channel FF

\[ \theta \rightarrow i\pi - \theta \]

\[ \hat{F}(\theta) \equiv \langle a \mid \varphi(0) \mid K_{ab}^\pm(\theta_1)K_{ba}^\pm(\theta_2) \rangle \]
If the resonance angle is within the physical strip, the masses of the bound states are

\[
m_n = 2M_{cl} \sin \frac{u_n}{2}
\]
Structure of the poles

\[ \xi_a = \frac{\omega_a}{M} \]

\[ \phi(x) = \phi_a + \sum_{n} b_n \exp\left(-n \xi_a x\right) \]
Poles in its Fourier transform
Most general expression of semiclassical Form Factors

\[ \varphi(x) \approx \sum_n b_n \exp(-n \xi_a x) \]

\[ F^G(k) = \int da \ e^{ika} G[\varphi(a)] = \sum_n \frac{g_n}{ik + n\xi} \]

The poles are fixed. The information on the operator G enters only in the coefficients \( g_n \)
Mass of the bound states

\[ m^B_a(n) = 2M \sin \frac{n \pi \xi_a}{2} \]

\[ n = 1, 2, \ldots \left[ \frac{1}{\xi_a} \right] \]
However, not all of them are necessarily stable!

\[ m_k > 2m_1 \]

\[ k = 3, 4, 5, \ldots \]

Conclusion: In a non-integrable QFT the number of stable particles around each vacuum can be at most \( 2^{m_k} \).
$\Phi^4$ in its broken phase

\[ U(\Phi) = \frac{\lambda}{4} \left( \Phi^2 - \frac{\mu^2}{\lambda} \right)^2 \]

\[ \xi = \frac{3\lambda}{2\pi\mu^2} \]

\[ \langle a \mid \Phi(0) \mid K_{-a,a}(\theta_1)K_{a,-a}(\theta_2) \rangle = \frac{1}{\sinh \frac{i\pi - \theta}{\xi}} \]

If $\xi > 1$, i.e. $\frac{\lambda}{\mu^2} > \frac{2\pi}{3}$, no bound states
$\Phi^4$ in its broken phase

$$U(\Phi) = \frac{\lambda}{4} \left( \Phi^2 - \frac{\mu^2}{\lambda} \right)^2$$

$$\xi = \frac{3\lambda}{2\pi\mu^2}$$

$$\langle a \mid \Phi(0) \mid K_{-a,a}(\theta_1)K_{a,-a}(\theta_2) \rangle = \frac{1}{\sinh \frac{i\pi - \theta}{\xi}}$$

If $\xi < 1$ there are instead bound states, only the first 2 stable
Asymmetric wells

$$U(\varphi) = \frac{\lambda}{2} \left( (\varphi + a)^2 (\varphi - b)^2 (\varphi^2 + c) \right)$$

$$\omega_0 \propto \sqrt{a^2 + c}$$  $$\omega_1 \propto \sqrt{b^2 + c}$$
Poles and bound states

\[ |K(\theta_1)\bar{K}(\theta_2)| \]

\[ |\bar{K}(\theta_1)K(\theta_2)| \]

Mass of the kink

\[ M = \frac{\lambda^2}{m^4} \]

If \( \omega_0 < \frac{\lambda^2}{m^4} < \frac{1}{\omega_1} \)

the first pole is out the physical strip but the second is inside!
Truncated Conformal Space Approach

Lassig, GM, Cardy;
Colomo, GM, Coser...
Truncated conformal space approach for 2D Landau–Ginzburg theories

A Coser1, M Beng1, G P Brandino2, R M Konik3 and G Mussardo1

1 SISSA—International School for Advanced Studies and INFN, Sezione di Trieste, Via Bonomea 265, I-34136 Trieste, Italy
2 Institute for Theoretical Physics, University of Amsterdam, Science Park 904, Postbox 94485, 1090 GL Amsterdam, The Netherlands
3 CIPAM, Department of H tod, 734, Brookhaven National Laboratory, Upton, NY 11973-4000, USA
4 The Abdus Salam International Centre of Theoretical Physics, 34100 Trieste, Italy

E-mail: acoser@sissa.it

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Abstract. We study the spectrum of Landau–Ginzburg theories in 1 + 1 dimensions using the truncated conformal space approach employing a compactified boson. We study these theories both in their broken and unbroken phases. We first demonstrate that we can reproduce the expected spectrum of a $\Phi^4$ theory (i.e. a free massive boson) in this framework. We then turn to $\Phi^4$ in its unbroken phase and compare our numerical results with the predictions of two-loop perturbation theory, finding excellent agreement. We then analyze the broken phase of $\Phi^4$ where kink excitations together with their bound states are present. We confirm the semiclassical predictions for this model on the number of stable kink-antikink bound states. We also test the semiclassics in the double well phase of $\Phi^4$ Landau–Ginzburg theory, again finding agreement.

Keywords: conformal field theory (theory), other numerical approaches, quantum phase transitions (theory)

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Close to Integrability
Form Factor Perturbation Theory

\[ A = A_{int} + g \int d^2 x \Phi(x) \]
Form Factor Perturbation Theory

(Delfino, Simonetti, G.M.; Delfino, G.M.; Controzzi, G.M.;...)

\[ \text{Diagram} \]
Particularly interesting is the case when we can interchange the pattern of integrability breaking.
Models analyzed by Form Factor Perturbation Theory

Multi-frequency Sine-Gordon (Delfino, GM)

O(3) sigma model with theta term (Controzzi, GM)

Tricritical Ising model with several couplings (Fioravanti, Simon, GM)

SUSY with metastable vacuum state (GM)

Tricks to obtain an infinite number of identities between FF
Main advantages of Form Factor Perturbation Theory

• It gives reason of the confinement of the particles, relating it to the semi-local property of the perturbing operator

• When the operator is local, the first order correctly captures the perturbed values (as shown, for instance, by comparison with numerical simulation)

   Tricks: universal ratios

Main disadvantages

• Quite difficult to go to higher orders
Processes analyzed hereafter:

- Mass corrections and kink confinement
- Decay processes of unstable (higher mass) particles
Mass correction

\[ \langle 0 | \varphi | A(\beta_1) A(\beta_2) \rangle \]

\[ F_2^\varphi(\beta) = \frac{R}{(\beta - i\pi)} + \text{reg} \]
\[ \delta m^2 \equiv g < A(\beta) | \varphi | A(\beta) > = g F_2^\varphi (i\pi) \]

\[ F_2^\varphi (\beta) = \frac{R}{(\beta - i\pi)} + \text{reg} \]
$\delta m^2 \equiv g \quad < A(\beta) | \varphi | A(\beta) > = g F_2^\phi (i\pi)$

$R \equiv -i \ res_{\beta=i\pi} F_2^\phi (\beta) = \left(1 - e^{2\pi i\gamma}\right) < 0 | \phi | 0 >$

- If $\varphi(x)$ is a local field ($\gamma=0$), $R = 0$: ADIABATIC SHIFT
- If $\varphi(x)$ is a non-local field ($\gamma\neq0$), $R \neq 0$: CONFINEMENT
Decay process of higher mass particles

\[ \Gamma \approx \frac{g^2}{2M} \left| \langle M | \varphi(0) | m_1(p)m_2(-p) \rangle \right|^2 \]

\[ |p| = \sqrt{[M^2 - (m_1 - m_2)^2] [M^2 - (m_1 + m_2)^2]} \]

(Fermi Golden Rule)

(Delfino, Grinza, GM)
Ising Field Theory

Scaling limit of 2-d Ising Model in a magnetic field and away from the critical temperature

\[ A = A_{\text{CFT}} + (T - T_c) \int d^2 x \epsilon(x) + h \int d^2 x \sigma(x) \]

(Delfino, Simonetti, GM)
(Fonseca, Zamolodchikov)
(Grinza, Delfino, GM)
Parameter of the Model

\[ \chi = \tau |h|^{-8/15} \]

labels the RG trajectories
$CFT$
Particular (integrable) points

- $h=0$ \rightarrow Massive free Majorana fermion
- $T=T_c$ \rightarrow Massive theory with 8 particles
Mass Spectrum

- $h=0$  Free neutral fermion

$T > T_c$: ordinary particle

$T < T_c$: kink and anti-kink
Kinks are confined.

\[ H = \int dx \left[ \frac{1}{2} (\nabla \phi)^2 + V(\phi) \right] \]

\[ H[\text{\textbullet\textbullet\textbullet}] = \infty \]

Kinks are confined.
In the familiar Ising model...

Kink excitations

Absent in the perturbed theory

\[
\langle 0 | \sigma(0) | K(\theta_1) K(\theta_2) \rangle = \tanh \frac{\theta_1 - \theta_2}{2}
\]
Kink excitations

\[ \delta m^2 \sim F(i\pi) = \infty \]
Kinks are confined. Spectrum given by kink-antikink bound states of mass given by

\[
H = \int dx \left[ \frac{1}{2} (\nabla \phi)^2 + V(\phi) \right]
\]

\[
(2 + h^{2/3} \gamma^{2/3}_k) m
\]

\[
J_{1/3} \left( \frac{1}{3} \gamma_k \right) + J_{-1/3} \left( \frac{1}{3} \gamma_k \right) = 0
\]
The number of stable particles decreases as $\eta$ increases, until only one is stable at $\eta \to +\infty$.
Namely, the plane is partitioned in sectors, for which there are a certain number of stable particles.

The particle $m_k$ becomes then unstable for $\chi > \chi_k$.
Mass spectrum in the vicinity of the Magnetic Axis with temperature different from the critical value

(Delfino, Simonetti, GM)

\[
\frac{\delta m_1}{\delta m_k} = 0.8616
\]

\[
\frac{\delta m_2}{\delta m_1} = -0.0558m_1
\]

\[
\frac{\delta m_3}{\delta m_1} = 1.508
\]
Decay channels of higher mass particles

\[ A_4 \rightarrow A_1 A_1 \]

\[ A_5 \rightarrow A_1 A_1 , A_1 A_2 \]

\[ A_6 \rightarrow A_1 A_1 , A_1 A_2 , A_1 A_3 \]

\[ f_{abc} \]
Decay widths of higher particles

\[ \frac{\Gamma_4}{\Gamma_5} = 4.287... \]

• Alias, the particle with higher mass lives 4 time longer than the one with lower mass!

• Branching ratios of the decay of \( A_5 \)

\( A_5 \rightarrow A_1 A_1 \)  \hspace{1cm} 47\%

\( A_5 \rightarrow A_1 A_2 \)  \hspace{1cm} 53\%
What is the fate of the various topological excitations?!

We expect their confinement (as the quarks) and to see just their neutral bound states ("mesons")
Fragility of the kink in purely bosonic theories

\[ L = \frac{1}{2} \left( \partial_\mu \Phi \right)^2 + \gamma \cos \beta \Phi \]
Fragility of the kink in purely bosonic theories

\[ L = \frac{1}{2} \left( \partial_\mu \Phi \right)^2 + \gamma \cos \beta \Phi + \eta \cos \omega \Phi \]
\[ \delta m^2 \equiv g < A(\beta) | \varphi | A(\beta) > = g F_2^\varphi (i\pi) \]

\[ R \equiv -i \ \text{res}_{\beta=i\pi} F_2^\varphi (\beta) = \left(1 - e^{2\pi i \gamma}\right) < 0 | \phi | 0 > \]

- If \( \varphi(x) \) is a local field (\( \gamma=0 \)), \( R = 0 \): ADIABATIC SHIFT
- If \( \varphi(x) \) is a non-local field (\( \gamma \neq 0 \)), \( R \neq 0 \): CONFINEMENT
Semi-local index of vertex operators in the SG model

\[ L = \frac{1}{2} (\partial \varphi)^2 - \lambda \cos \beta \varphi \]

\[ O = \cos \alpha \varphi \quad \gamma = \frac{\alpha}{\beta} \]

\[ \text{Res}_{\beta = i\pi} F^O_2 = [1 - \cos(2\pi \gamma)] \langle 0 | \mathcal{O}(0) | 0 \rangle \]

• If \( \gamma \) is irrational, no solitons survive

• If \( \gamma = \frac{m}{n} \) the original soliton confines but “packages” made of \( n \) original solitons survive as stable excitations
While kink excitations are rather fragile objects in purely bosonic theory, they can be instead stable excitations in integrability breaking that takes place in SUSY theories.

“Fermions stabilize the kinks”
\[ A = \int dxd\theta \left[ \frac{1}{4} (D\Phi)(D\Phi) - \cos \Phi - \lambda \cos(\omega \Phi) \right] \]

\[ \omega = \frac{p}{q} \]

\[ [W'(\varphi)]^2 = [\sin \varphi + \lambda \sin(\omega \varphi)]^2 \]

For all values of \( \lambda \) the origin is always a zero, i.e. SUSY exact

At lowest order, the vacua of the original potential continue to remain degenerate.
For all values of $\lambda$, the origin is always a zero, i.e. SUSY exact.

At lowest order, the vacua of the original potential continue to remain degenerate. The kinks are **stable** at weak coupling!
Multi-frequency Super Sine-Gordon (GM, JHEP)

\[
A = \int dx d\theta \left[ \frac{1}{4} (\overline{D}\Phi)(D\Phi) - \cos \Phi - \lambda \cos(\omega \Phi) \right]
\]

\[
\omega = \frac{p}{q}
\]

\[
[W'(\varphi)]^2 = [\sin \varphi + \lambda \sin(\omega \varphi)]^2
\]

- For all values of \( \lambda \) the origin is always a zero, i.e. SUSY exact.
- At lowest order, the vacua of the original potential continue to remain degenerate. The kinks are **stable** at weak coupling!

**Form Factor Perturbation Theory**

\[
Y = \sin \varphi \sin \omega \varphi + \omega \psi \bar{\psi} \cos \omega \varphi
\]

\[
\text{Re} s_{\theta = \pm i\pi} F^Y_s s (\theta) = \left[ 1 + \cos(\pi \varphi) \right] < 0 | Y | 0 > = 0
\]
What happens at strong coupling?

\[ (W'(\varphi))^2 = \left[ \sin \varphi + \lambda \sin \left( \frac{p}{q} \varphi \right) \right]^2 \]

\[ N(\lambda) = \text{Number of zeros in } (0, 2\pi q) \]

\[ N(0) = 2q + 1 \]
\[ N(\infty) = 2p + 1 \]

\[ \Delta N = 2(q - p) \]
What happens at strong coupling?

\[(W'(\varphi))^2 = \left[ \sin \varphi + \lambda \sin \left( \frac{p}{q} \varphi \right) \right]^2\]

\[N(\lambda) = \text{Number of zeros in } (0, 2\pi q)\]

\[N(0) = 2q + 1 \quad N(\infty) = 2p + 1\]

\[\Delta N = 2(q - p)\]

- Since the kinks owe their existence to the zeros, a variation of their number implies that some of them should disappear.

- For their topological nature, the disappearance of a kink signals a phase transition.
When \((q-p) = \text{even}\), there will be a sequence of phase transitions that will recall the one of Tricritical Ising -> Ising.

**Local SUSY breaking, with meta-stable vacua**
When \((q-p) = \text{odd}\), there will be, in addition to a sequence of phase transitions TIM -> Ising, also a phase transition that will recall the one of the \text{gaussian model}.

In this case, there is a vacuum where \text{SUSY is exact} before and after the phase transition.
Conclusions and Perspectives

• There is a rather robust knowledge on how to control and predict the spectrum in a generic 2-dim QFT

• In the perspective to explore their experimental study, particularly interesting is the pattern of kink excitations in multi Sine-Gordon theories

• Equally interesting is the possibility to realize experimentally SUSY theories through mixture and to check in particular the persistency of the kink excitations