How to control the breaking of integrability

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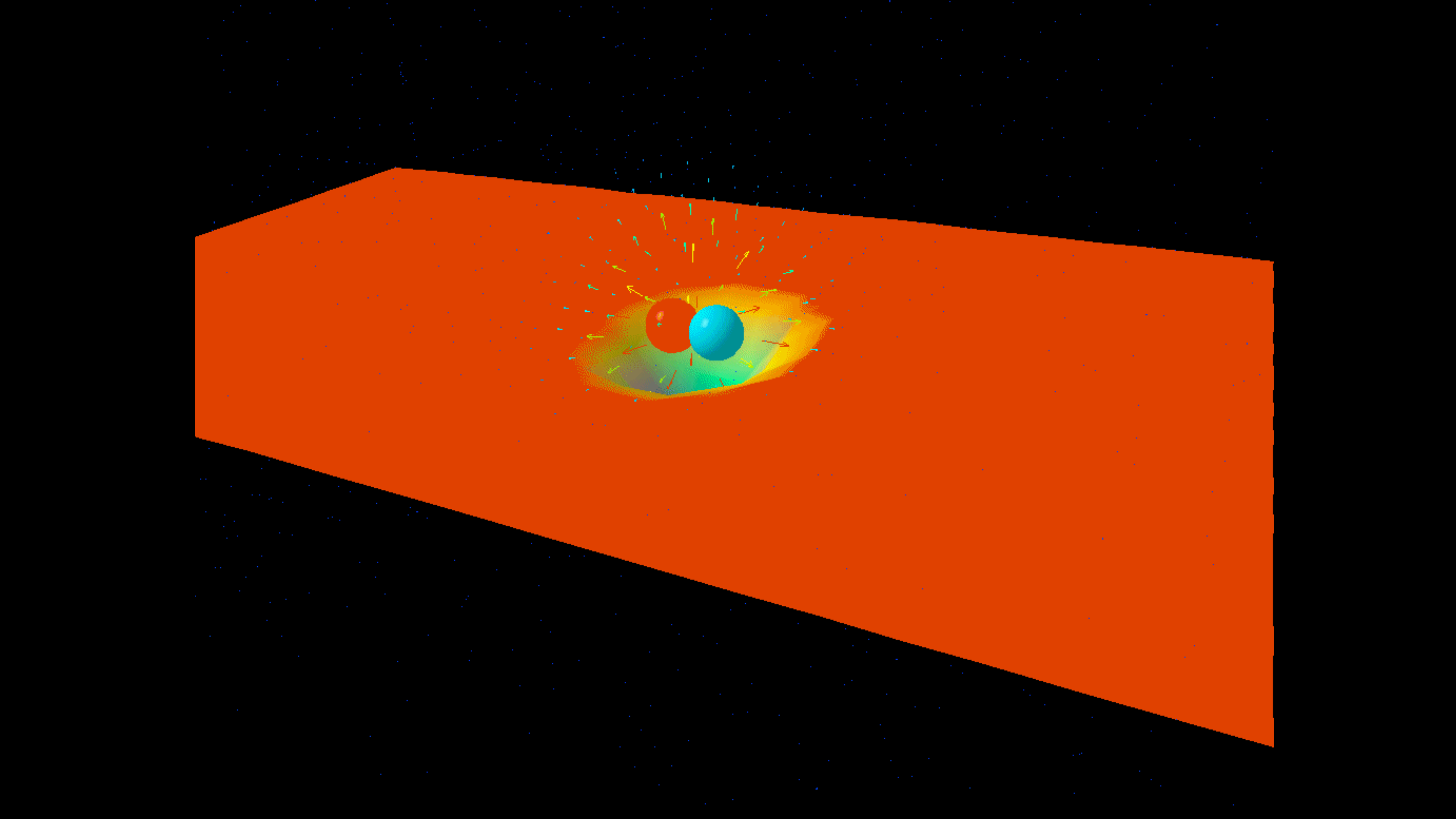
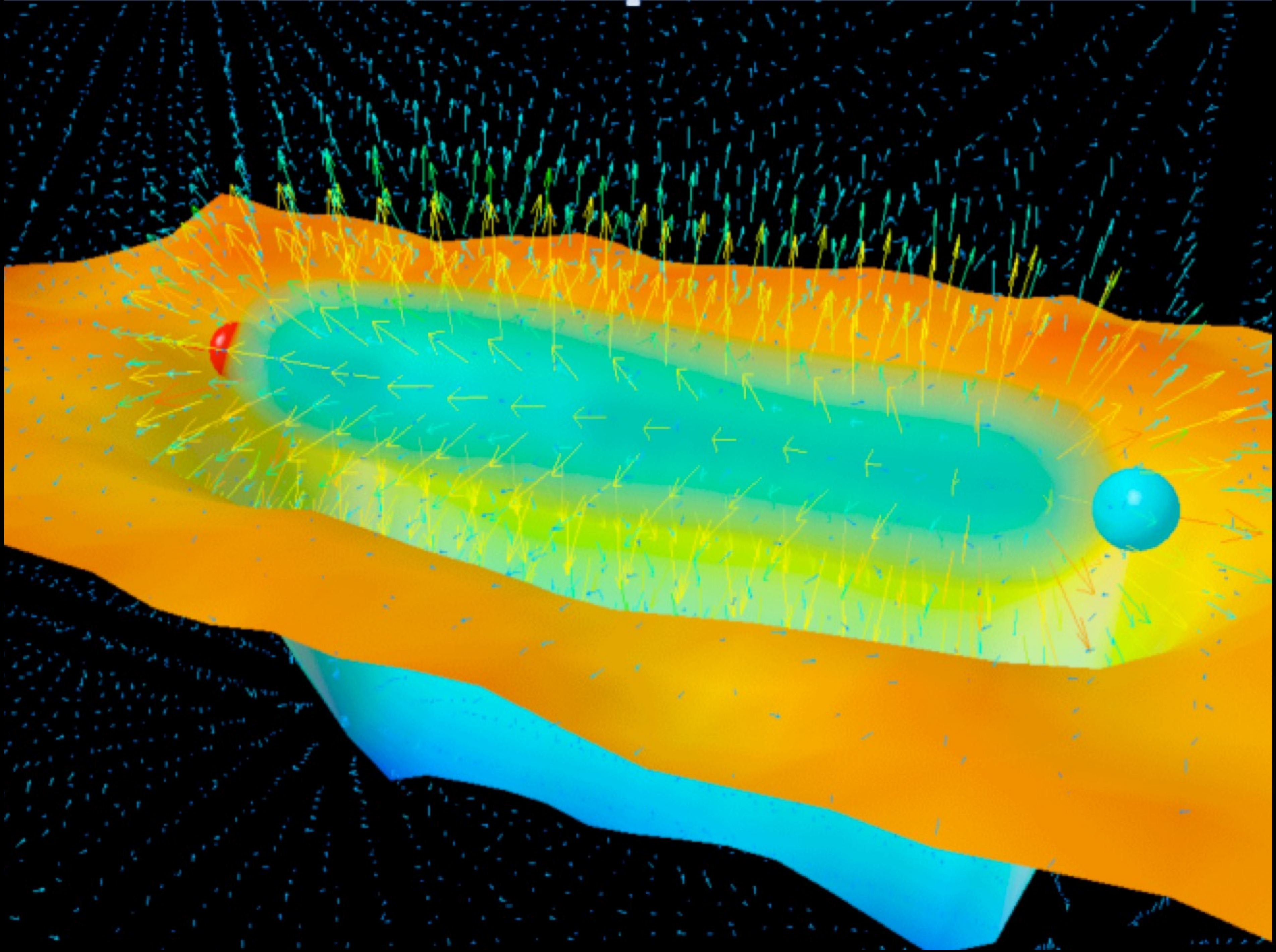
Topics of the seminar

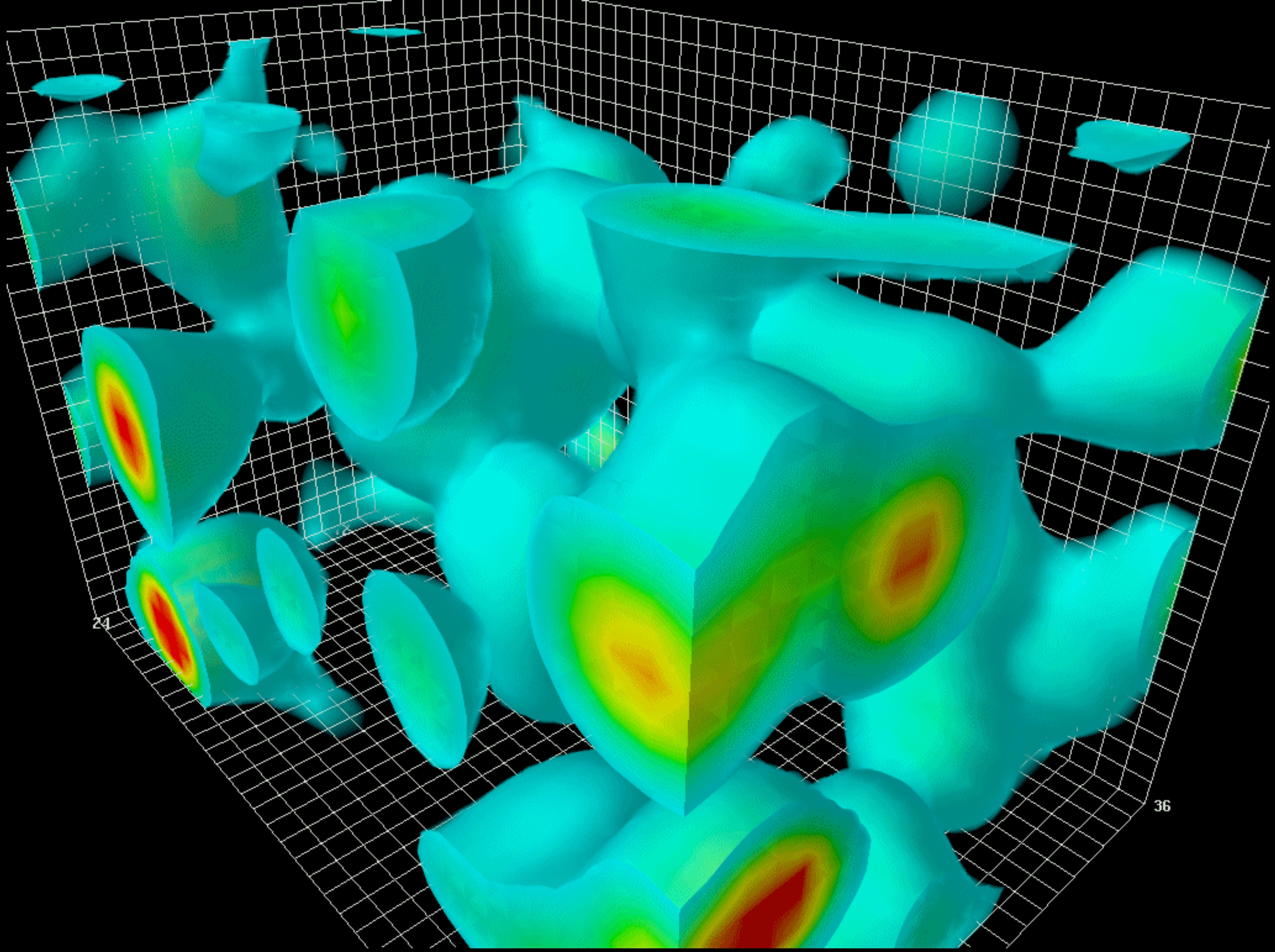
- The problem of the spectrum
- The art of integrable models
  - (i) Some renowned examples
  - (ii) Experimental signatures
- Breaking integrability
  - (i) Truncated Hamiltonian approaches
  - (ii) Semi-classical methods
  - (iii) Form Factor Perturbation Theory
- Confinement
  - (i) Fragility of the kinks
  - (ii) The touch of supersymmetry

On a general footing, determining the spectrum of a QFT is quite a difficult problem...

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \sum_c \bar{q}_c \left[i\gamma^\mu (\partial_\mu - igA_\mu) - m_c \right] q_c$$



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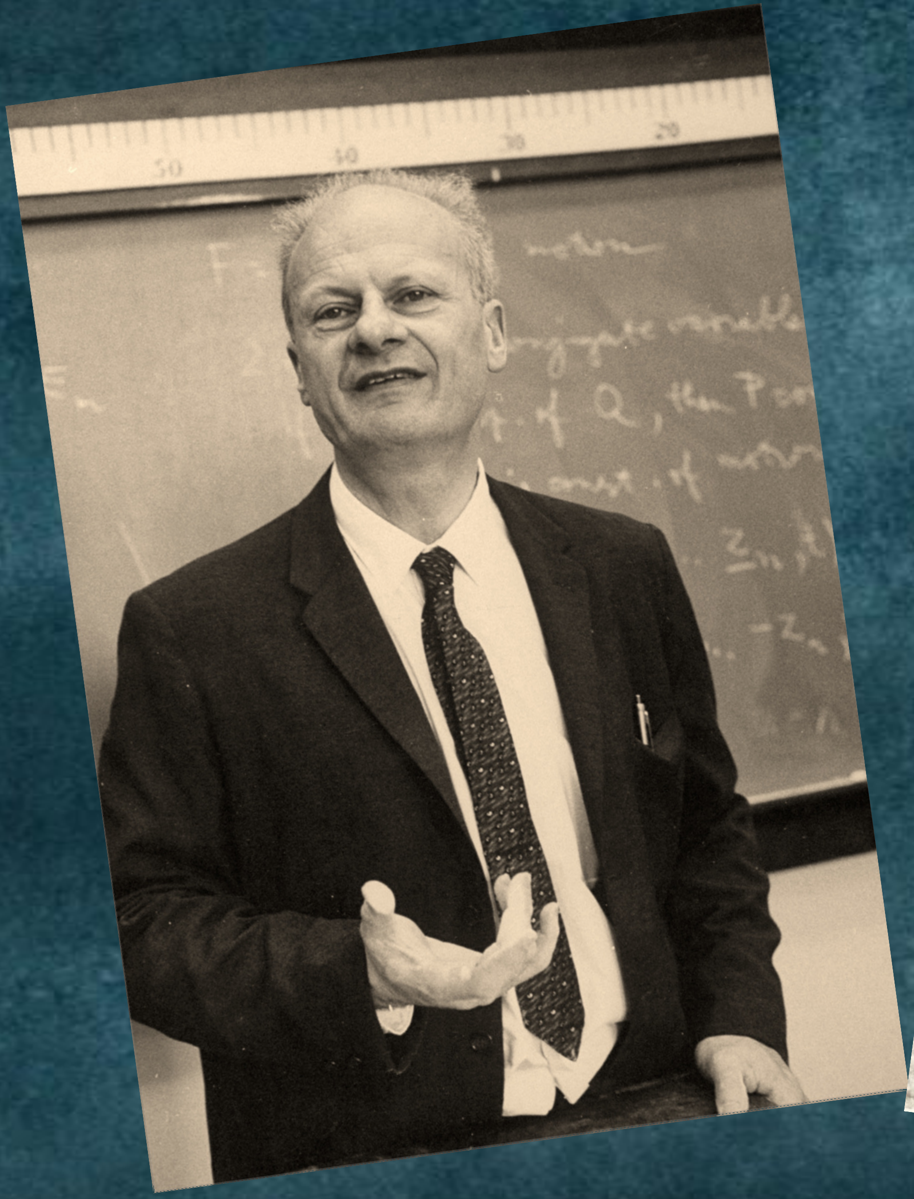
Few methods are usually available to study the dynamical spectrum (i.e. bound states) of a QFT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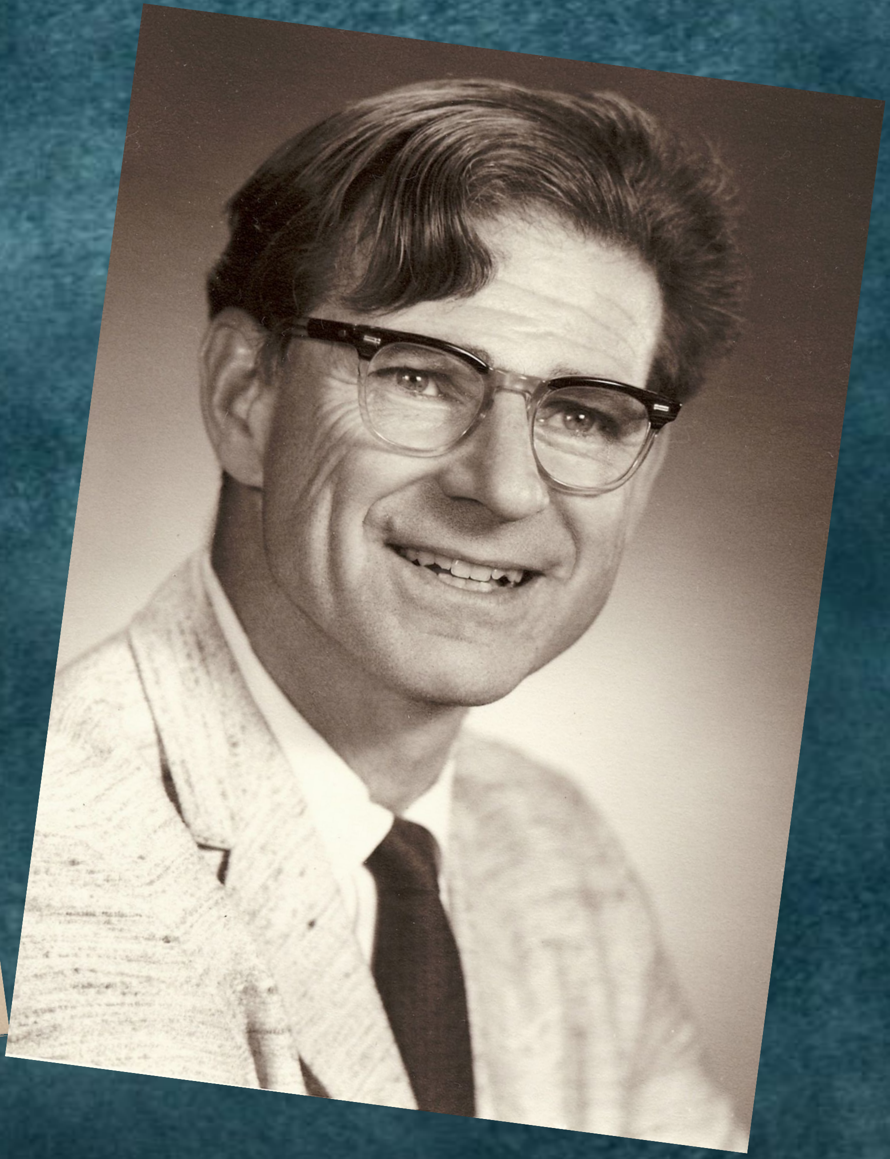
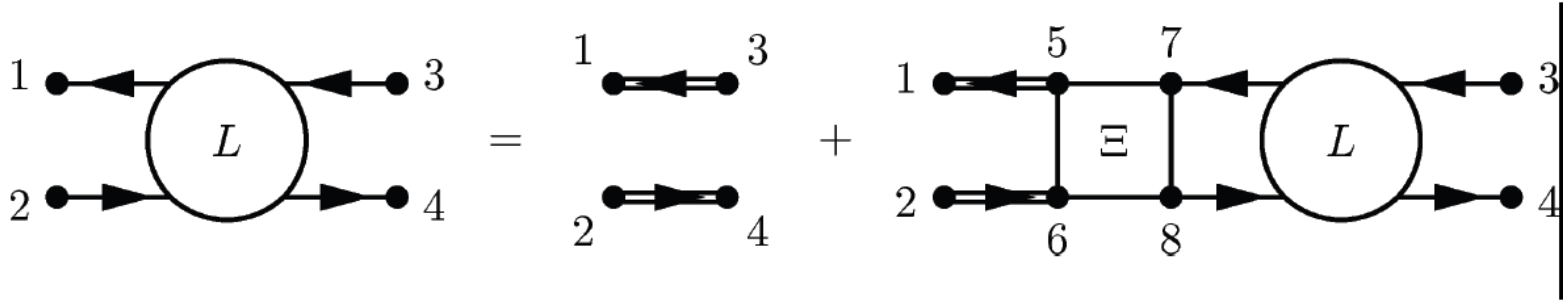
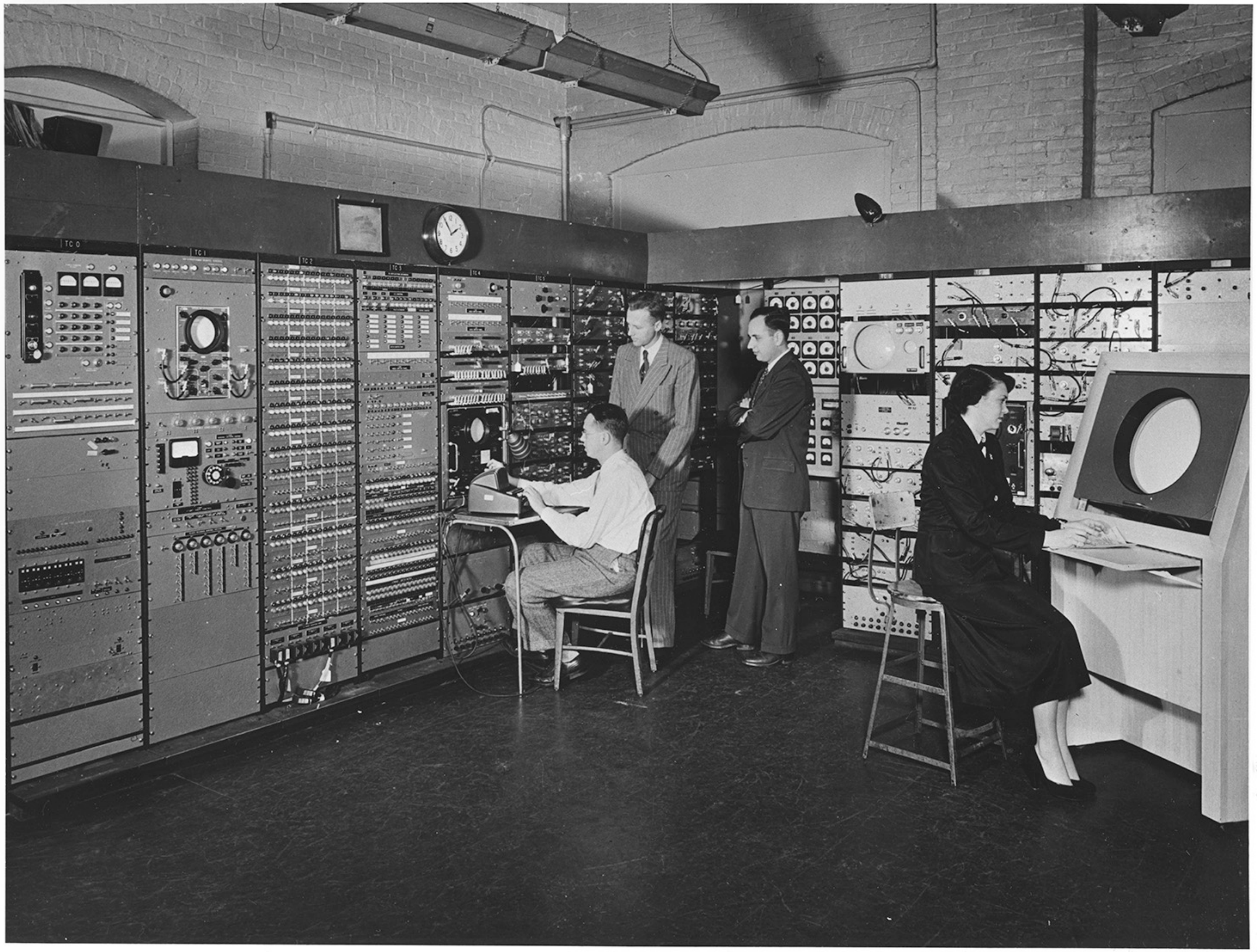
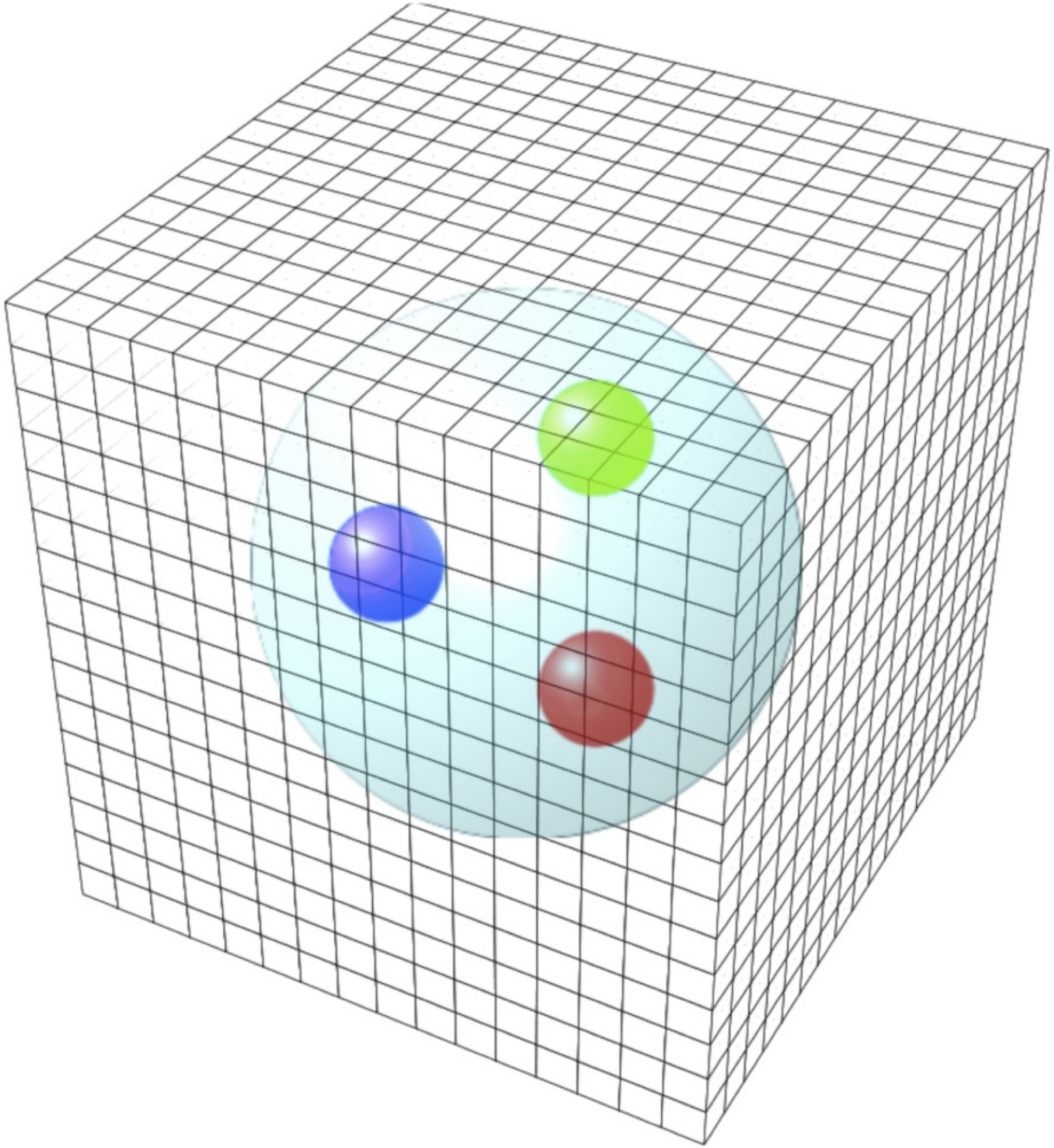
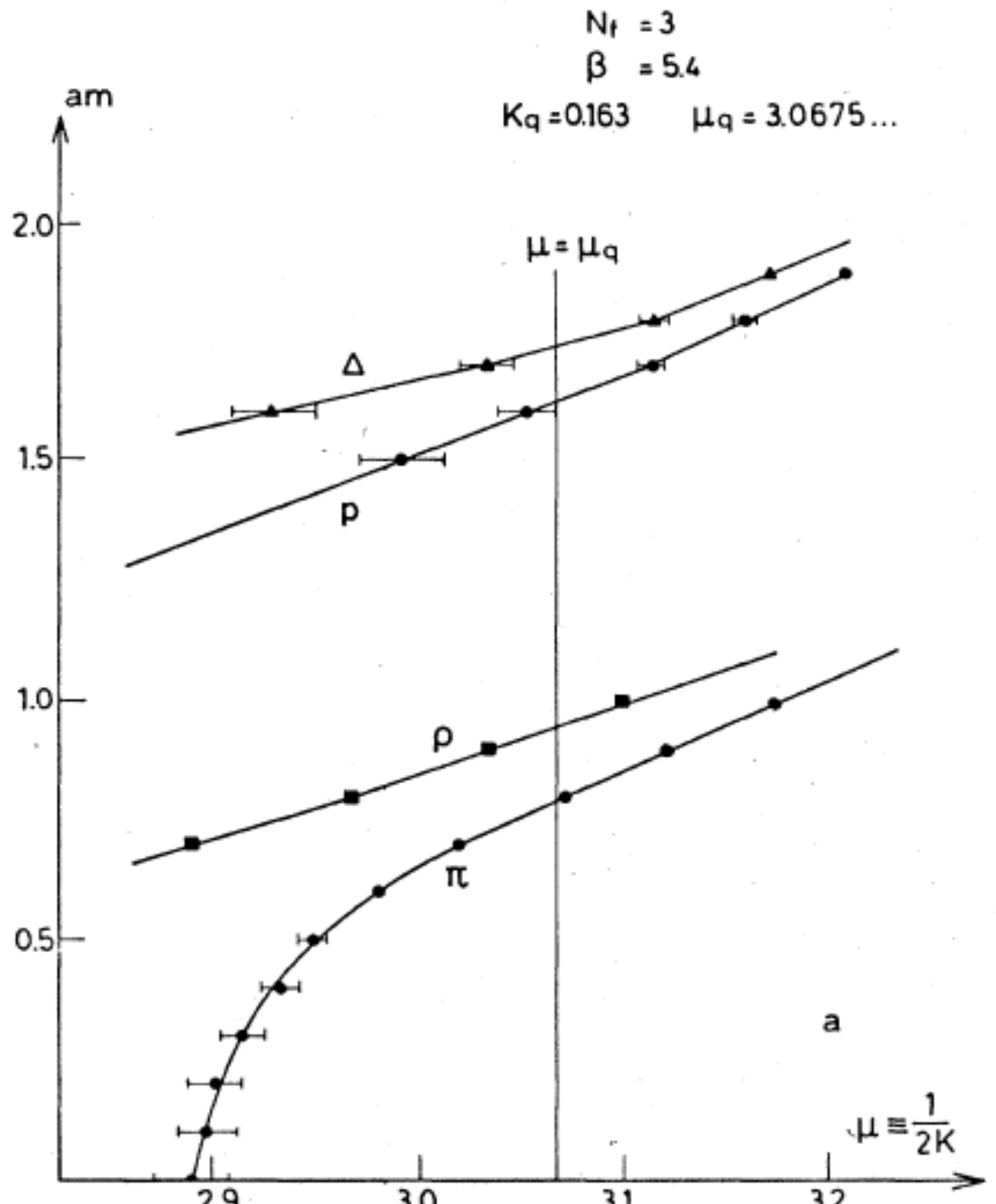




Bethe-Salpeter Equation (1951)

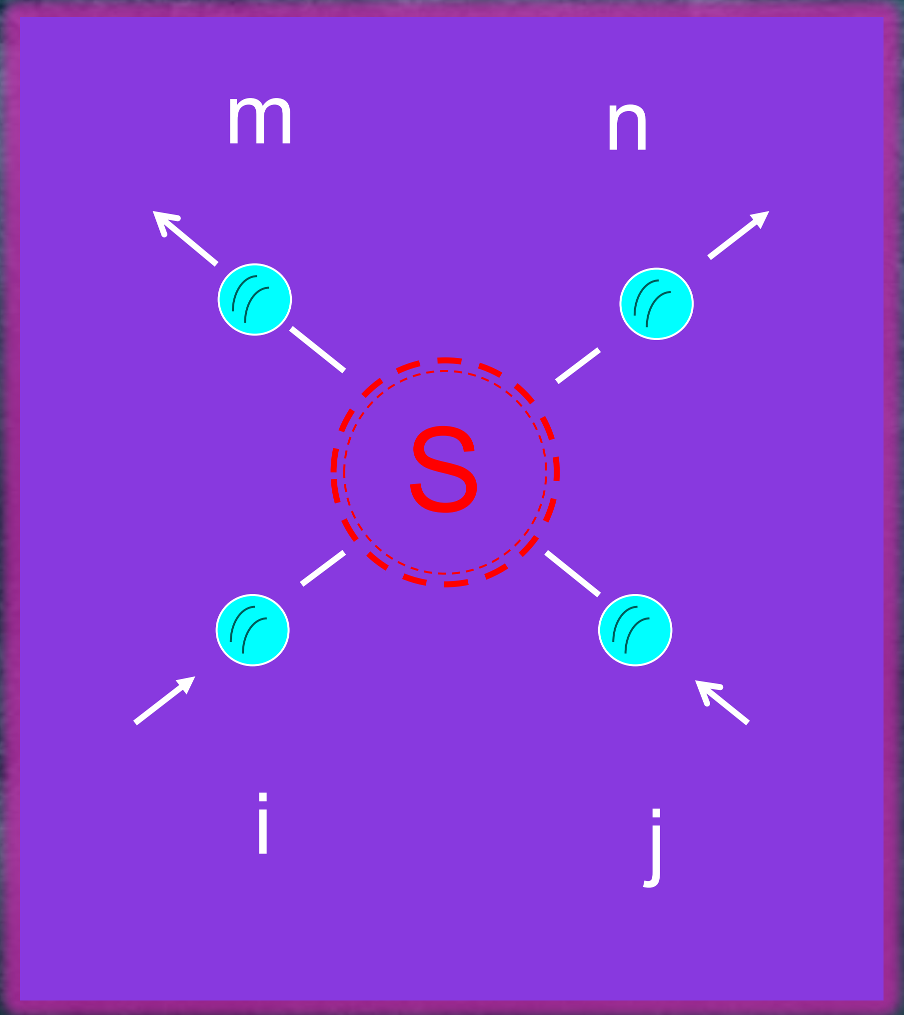






$N_f = 3$
$\beta = 5.4$
$K_q = 0.163$ $\mu_q = 3.0675...$
am
2.0
1.5
1.0
0.5
$\mu = \mu_q$
Δ
p
ρ
π
a
$\mu \equiv \frac{1}{2K}$
2.9
3.0
3.1
3.2

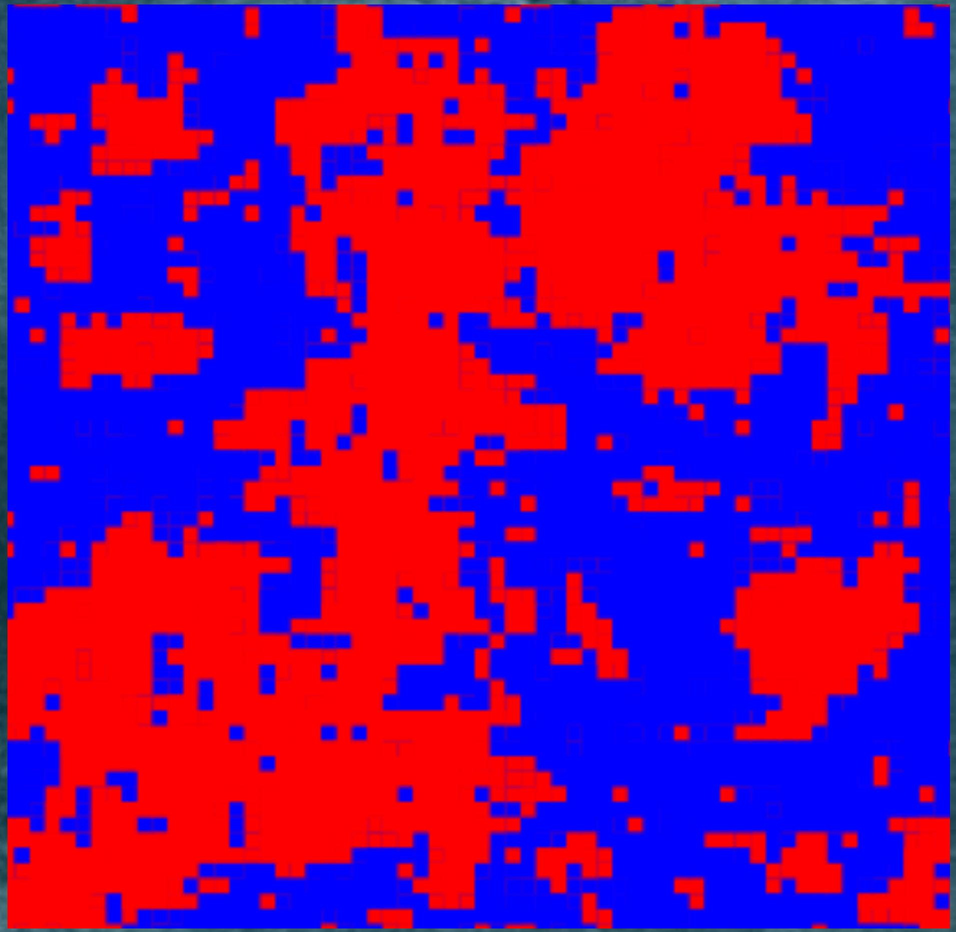
In two dimensions the exact spectrum can be extracted by looking at the poles of the exact S-matrix, when the system is integrable



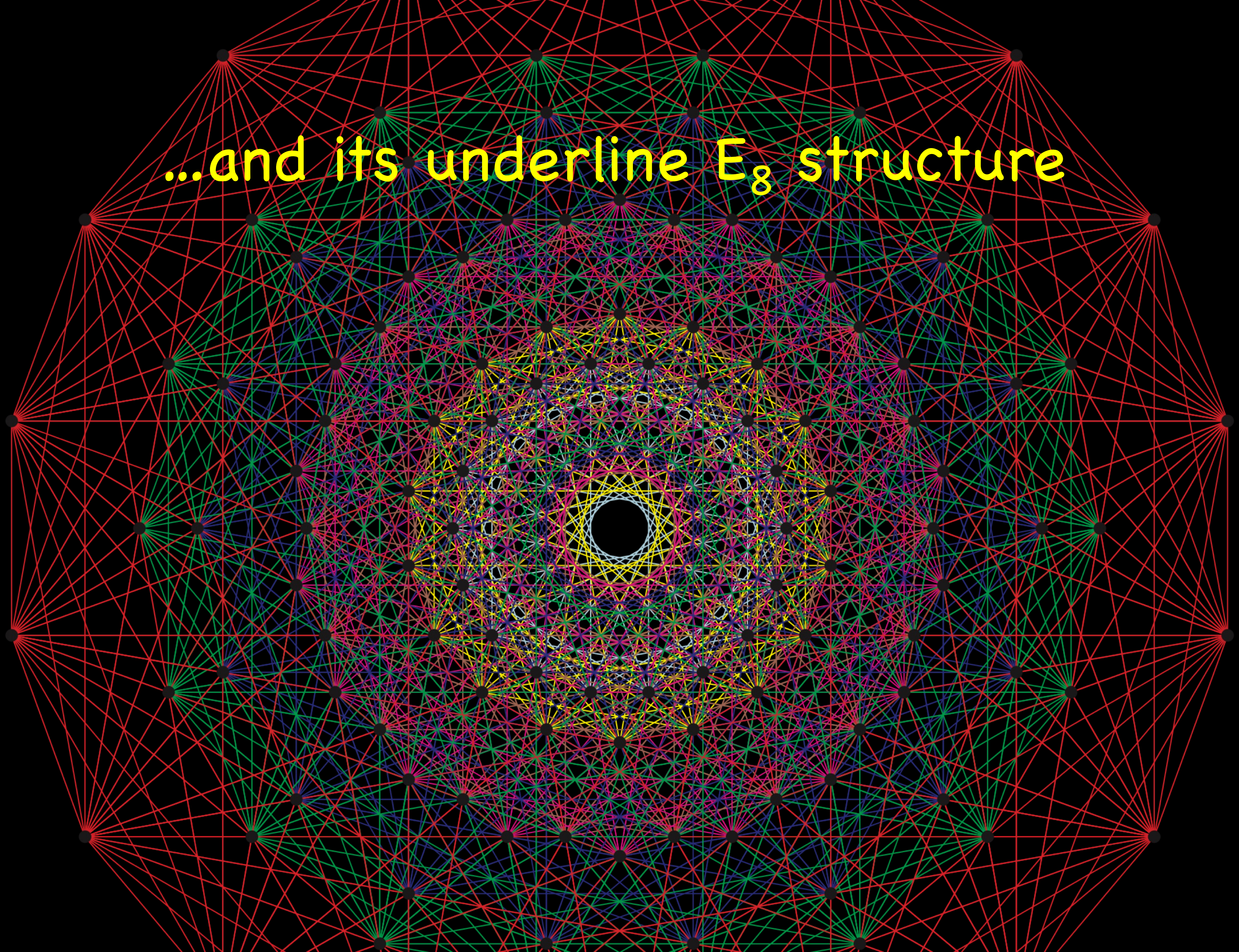
In two dimensions the exact spectrum can be extracted by looking at the poles of the exact S-matrix, when the system is integrable

(Sine-Gordon model, Ising model in a magnetic field, Yang-Lee model, Tricritical Ising Model, Potts model, etc.)

A remarkable example: 2d Ising model in a magnetic field at $T=T_c$ ...



…and its underline E_8 structure



International Journal of Modern Physics A, Vol. 4, No. 16 (1989) 4235–4248

INTEGRALS OF MOTION AND S-MATRIX OF THE (SCALED) $T = T_c$ ISING MODEL WITH MAGNETIC FIELD

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Received 24 January 1989

It is shown that the field theory describing the scaling limit of $T = T_c$ Ising model with nonzero magnetic field possesses a number of nontrivial local integrals of motion. The exact mass spectrum and S-matrix of this field theory is conjectured.

1. Introduction

Conformal field theory (CFT) and Integrable field theory (IFT) in two dimensions are two subjects which attracted much attention in the last years. The subjects seem to be deeply related. Very close mathematics is involved in treatment of both theories (see Ref. 1 and references therein). Also, the ultraviolet limit (and sometimes also the infrared limit[2,3]) of IFT is described by CFT and so the general IFT can be considered as the CFT perturbed by particular "integrable" relevant operator.[4,5] The most simple example is the scaling limit $T \to T_c$ of the Ising model (with zero magnetic field), which is well known to be the (certainly integrable) field theory of free massive Majorana fermions; it can be considered as $c = 1/2$ CFT perturbed by the spinless primary field $\varepsilon = \Phi_{(1,3)}$ ("energy density") having the conformal dimensions $(1/2, 1/2)$.[a]

In this paper we consider the same $c = 1/2$ CFT but now perturbed by the Z_2 odd primary operator $\sigma = \Phi_{(1,2)}$

$$H_{1/2}^{(1,2)} = H_{1/2} + h \int \sigma(x) d^2x \tag{1.1}$$

where $H_{1/2}$ is the Action (or Hamiltonian in statistical physics) of the $c = 1/2$ CFT and h is the (dimensional) constant. The field $\sigma(x)$ (which have the dimensions $(1/16, 1/16)$) is interpreted as the spin density in the critical $T = T_c$ Ising model and so the Hamiltonian (1.1) describes the scaling limit of $T = T_c$ Ising model with nonzero magnetic field h. It will be shown that the field theory (1.1) possesses several nontrivial local integrals of motion (IM) of the form

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[a] The algebraic structure of this off-critical IFT and its connection with the algebraic structure of CFT i...

ELSEVIER

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NUCLEAR PHYSICS B [FS]

The spin–spin correlation function in the two-dimensional Ising model in a magnetic field at $T = T_c$

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Abstract

The form-factor bootstrap approach is used to compute the exact contributions in the large-distance expansion of the correlation function $\langle\sigma(x)\sigma(0)\rangle$ of the two-dimensional Ising model in a magnetic field at $T = T_c$. The matrix elements of the magnetization operator $\sigma(x)$ present a rich analytic structure induced by the (multi-) scattering processes of the eight massive particles of the model. The spectral representation series has a fast rate of convergence and perfectly agrees with the numerical determination of the correlation function.

1. Introduction

Over the past few years, considerable progress has been made in the use of conformal invariance methods and scattering theory for the understanding of the critical points and the nearby scaling region of two-dimensional statistical models (see, for instance Refs. [1,2]). At the critical points, the correlation functions of the statistical models fall into a scale-invariant regime and their computation may be achieved by solving the linear differential equations obtained by the representation theory of the infinite-dimensional conformal symmetry [3]. The situation is different away from criticality. The scaling region may be described in terms of the relevant deformations of the fixed point actions. These deformations destroy the long-range fluctuations of the critical point and the associated quantum field theories are usually massive. If an infinite number of conserved charges survive the deformation of the critical point action, the corresponding QFT can be efficiently characterized (on-shell) by the relativistic scattering processes of

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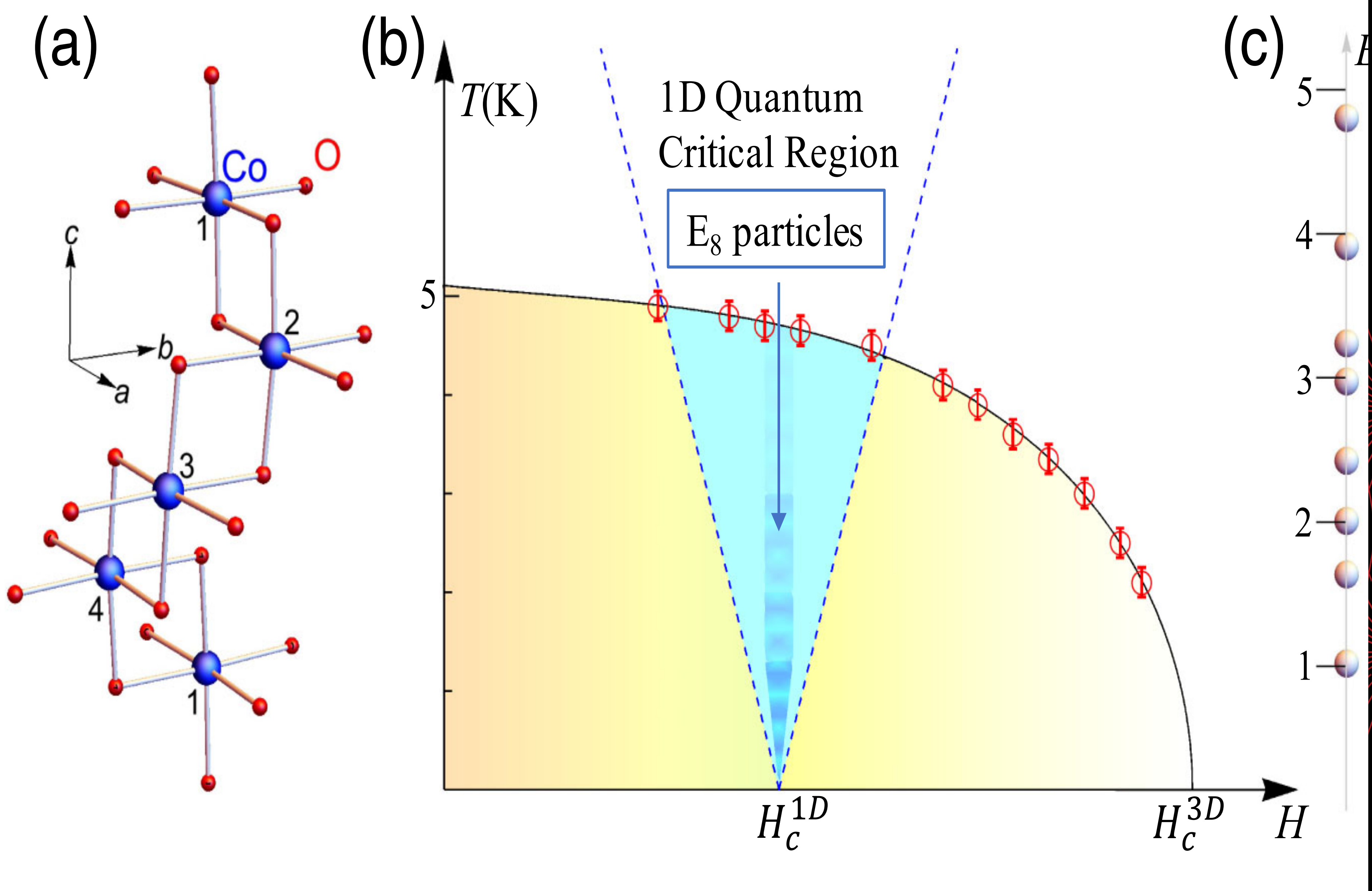
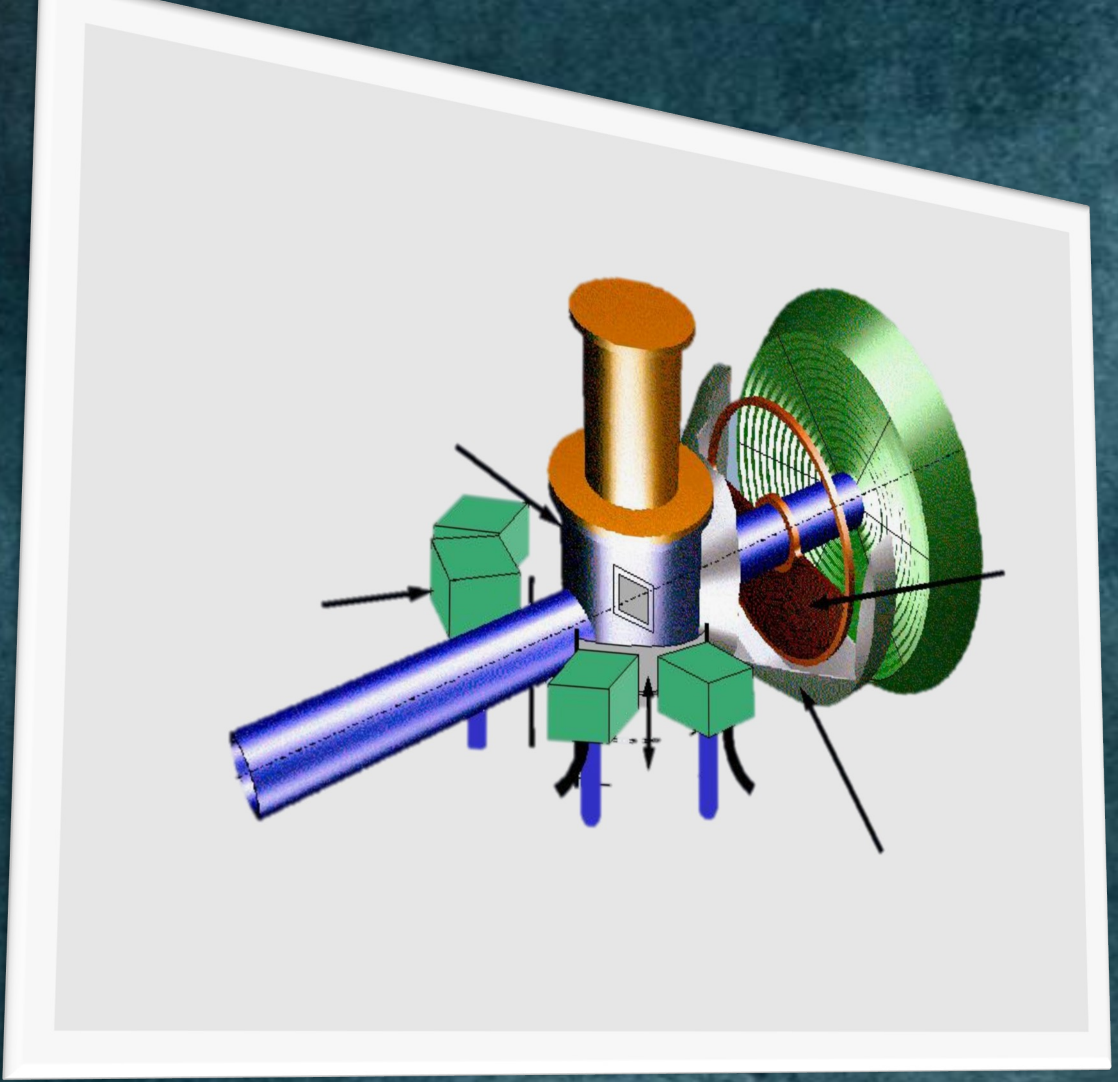
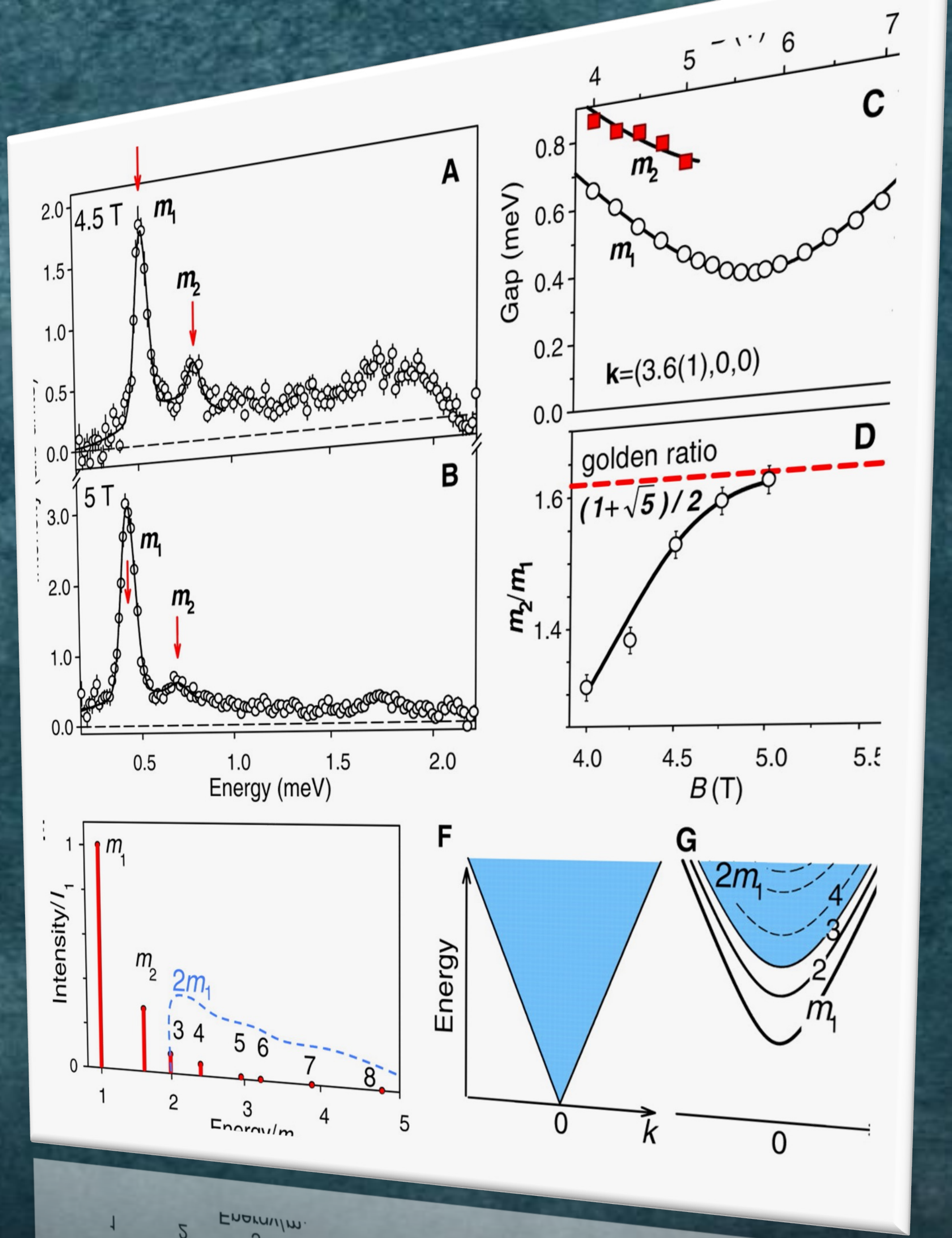


Spin-chain material $BaCo_2V_2O_8$





Coldea et al.
Science 2010

PHYSICAL REVIEW B **101**, 220411(R) (2020)

Rapid Communications

Observation of E_8 particles in an Ising chain antiferromagnet

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Near the transverse-field-induced quantum critical point of the Ising chain, an exotic dynamic spectrum consisting of exactly eight particles was predicted, which is uniquely described by an emergent quantum integrable field theory with the symmetry of the E_8 Lie algebra, but rarely explored experimentally. Here we use high-resolution terahertz spectroscopy to resolve quantum spin dynamics of the quasi-one-dimensional Ising antiferromagnet $BaCo_2V_2O_8$ in an applied transverse field. By comparing to an analytical calculation of the dynamical spin correlations, we identify E_8 particles as well as their two-particle excitations.

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Exotic states of matter, such as high-temperature superconductivity or magnonic Bose-Einstein condensation, can emerge in the vicinity of a quantum critical point [1], which identifies a zero-temperature phase transition tuned by an external parameter, e.g., chemical substitution or applied magnetic field [2,3]. Quantum critical points are often characterized by enhanced many-body fluctuations together with divergence of correlation length and complex emergent symmetry [1,4–8]; thus it is generally a formidable task to precisely describe the quantum many-body physics near a quantum critical point. Exactly solvable models play a crucial role in this regard, because a precise understanding of the quantum many-body physics can be gained by rigorously analyzing these models [4,6]. The one-dimensional (1D) spin-1/2 Ising model in a transverse magnetic field is such a paradigmatic example [1,4–9]. Considering only the exchange interaction between the nearest-neighbor spins on a chain [10,11], this model has been investigated most broadly in quantum magnetism, which provides deep insights into the fundamental aspects of the quantum many-body physics [1,6–8]. In particular, highly unconventional dynamic properties have been theoretically predicted to emerge near the transverse-field Ising quantum critical point, either for equilibrium states upon constant perturbations or for states far from equilibrium after a quantum quench (see, e.g., Refs. [12–18]). Moreover, the study of the transverse-field Ising quantum critical point is of importance also in the context of quantum information [5,8] and quantum simulation using ultracold atoms [19].

A remarkable prediction of an exotic dynamic spectrum was made three decades ago for the transverse-field Ising chain perturbed by a small longitudinal field [12]. It is described by the Hamiltonian

$$H = -J\sum_i S_i^z S_{i+1}^z - B_\perp \sum_i S_i^x - B_\parallel \sum_i S_i^z, \qquad (1)$$

with the x and z components S_i^x and S_i^z, respectively, of the spin-1/2 magnetic moment at the ith site on a 1D chain. The first term is the Ising term with the ferromagnetic exchange $J > 0$ between the nearest-neighbor spins. The second and third terms describe the interactions of the spins with the transverse field $B_\perp$ and the perturbative longitudinal field $B_\parallel$, respectively. Close to the transverse-field Ising quantum critical point [see Fig. 1(b)], the excitation spectrum of this model was predicted to be governed by a complex symmetry which is described by a quantum integrable field theory with the E_8 symmetry (an exceptional simple Lie algebra of rank 8) [12], which, however, is rarely explored experimentally. An analytical solution of the E_8 excitation spectrum delivered exactly eight particles ($\mathbf{m}_1$ to $\mathbf{m}_8$), the existence of which is uniquely determined by the specific ratios of their masses (Table I) with the lowest mass scaling with the perturbative longitudinal field; i.e., $\mathbf{m}_1 \propto |B_\parallel|^{8/15}$ [12]. Further analysis on the dynamic characteristics of the eight particles showed that the single-particle spectral weight decreases monotonically and drastically with increasing energy [Fig. 1(a)] [13,14]. Despite the apparent simplicity of the spin Hamiltonian in Eq. (1), an experimental realization of the E_8 spectrum, however, is very difficult, because several crucial criteria must be simultaneously fulfilled: one-dimensionality of spin interactions, strong Ising anisotropy, and a perturbative longitudinal field.

In this work, we use high-resolution terahertz (THz) spectroscopy to resolve E_8 particles in an antiferromagnetic Ising spin-chain material $BaCo_2V_2O_8$, where all the crucial criteria are found to be realized. By performing analytical

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E_8 Spectra of Quasi-One-Dimensional Antiferromagnet $BaCo_2V_2O_8$ under Transverse Field

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We report ^{51}V NMR and inelastic neutron scattering (INS) measurements on a quasi-1D antiferromagnet $BaCo_2V_2O_8$ under transverse field along the [010] direction. The scaling behavior of the spin-lattice relaxation rate above the Néel temperatures unveils a 1D quantum critical point (QCP) at $H_c^{1D} \approx 4.7$ T, which is masked by the 3D magnetic order. With the aid of accurate analytical analysis and numerical calculations, we show that the zone center INS spectrum at H_c^{1D} is precisely described by the pattern of the 1D quantum Ising model in a magnetic field, a class of universality described in terms of the exceptional E_8 Lie algebra. These excitations are nondiffusive over a certain field range when the system is away from the 1D QCP. Our results provide an unambiguous experimental realization of the massive E_8 phase in the compound, and open a new experimental route for exploring the dynamics of quantum integrable systems as well as physics beyond integrability.

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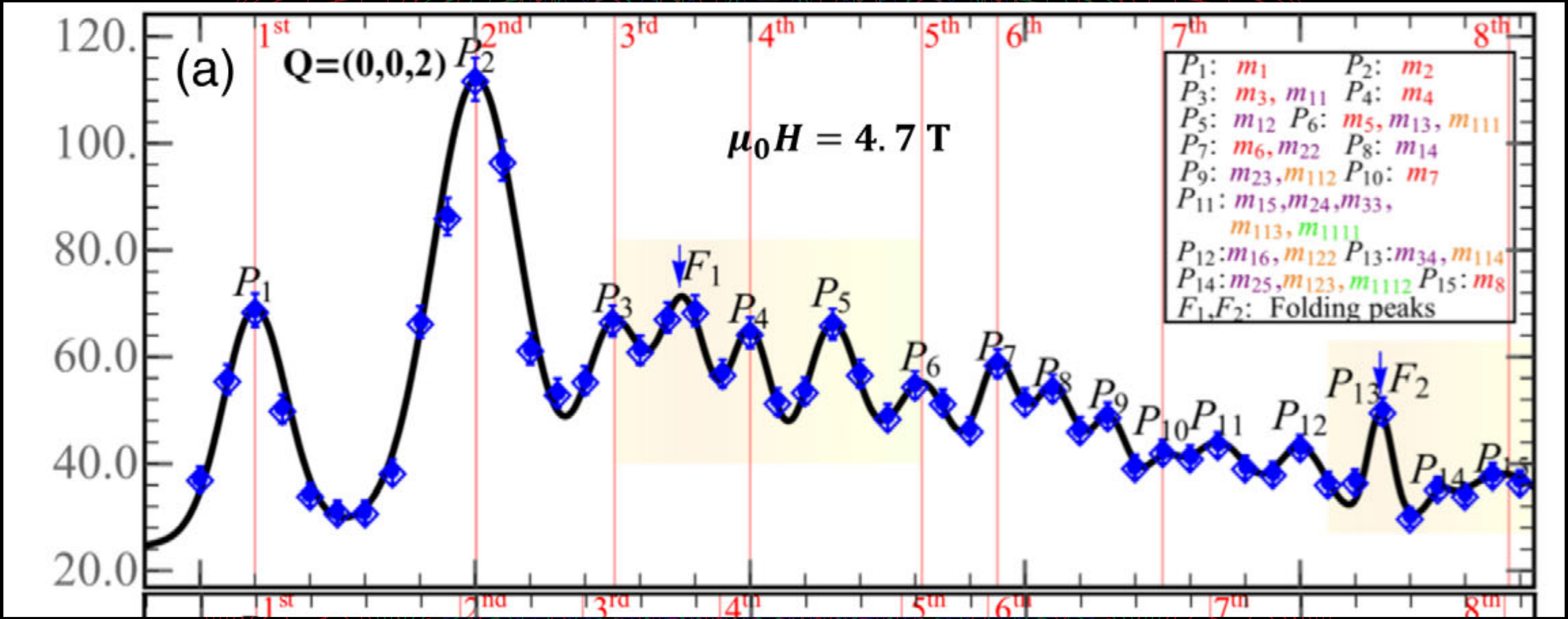
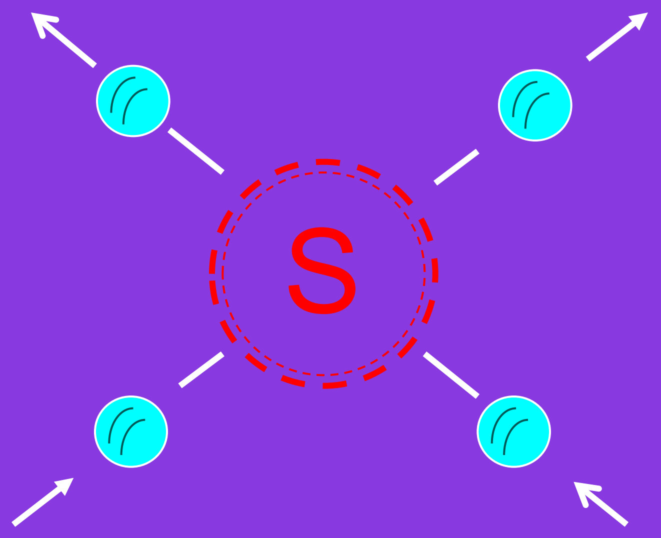
Strong fluctuations in the vicinity of quantum phase transitions can induce exotic ground states and excitations [1] such as unconventional quantum critical scalings [1,2], deconfined quantum critical points (QCPs) [3,4], and emergent enriched symmetries [5]. However, pursuing intrinsic features of these exotic states is a challenging quest, and only a few exactly solvable models provide significant insight. For example, an exotic spin liquid ground state can be characterized by the honeycomb Kitaev model [6] and stimulates serious hunting in materials [7]. Remarkably, an integrable model [8] emerges when the QCP of the paradigmatic 1D transverse-field Ising chain (TFIC) [1,2] is perturbed by a longitudinal magnetic field. The excitations of this model are beautifully characterized by the interplay of eight particles governed by the E_8 exceptional Lie algebra. This E_8 picture is a compelling pattern of the general class of the universality of the 1D TFIC once perturbed by a longitudinal magnetic field, as shown originally by Zamolodchikov [8]. Therefore, finding and exploring the E_8 physics in condensed matter systems will be a significant milestone for realizing analytically predicted emergent exotic excitations and will provide a manipulable platform for exploring quantum magnetism.

Compelling though it may be, the manifestation of this exotic E_8 physics can only be established via a dynamics study. In experiments, it is very challenging to accurately determine the location of a 1D QCP and resolve all the

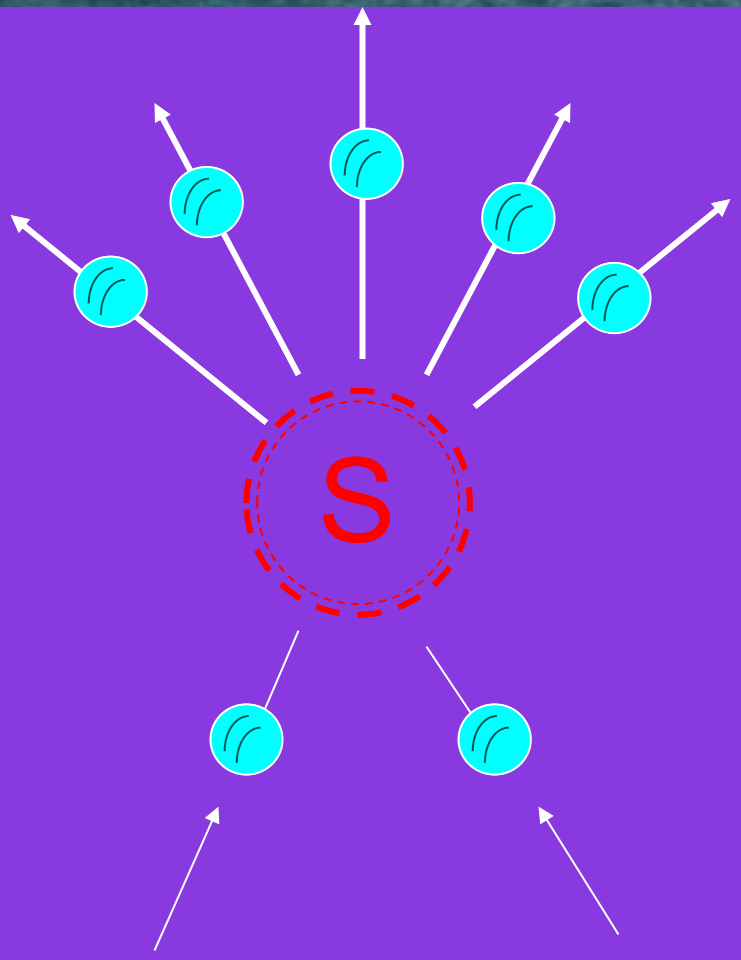


(a)
Q=(0,0,2)
$\mu_0 H$ = 4.7 T
1st
2nd
3rd
4th
5th
6th
7th
8th
120.
100.
80.0
60.0
40.0
20.0
P1
P2
P3
F1
P4
P5
P6
P7
P8
P9
P10
P11
P12
P13
F2
P14
P1: m1 P2: m2
P3: m3, m11 P4: m4
P5: m12 P6: m5, m13, m111
P7: m6, m22 P8: m14
P9: m23, m112 P10: m7
P11: m15, m24, m33, m113, m1111
P12: m16, m122 P13: m34, m114
P14: m25, m123, m1112 P15: m8
F1, F2: Folding peaks

Integrable vs NonIntegrable QFT

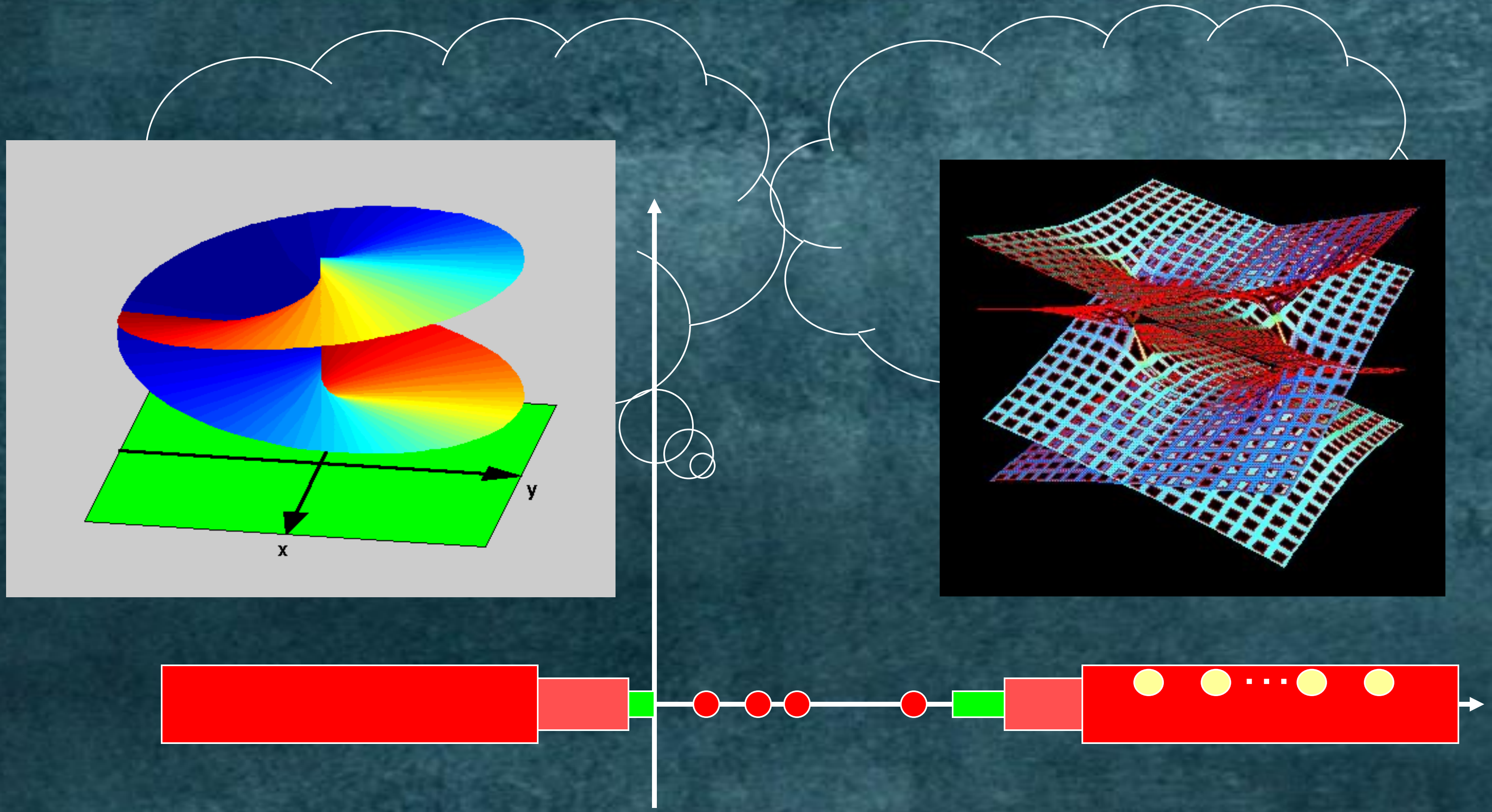


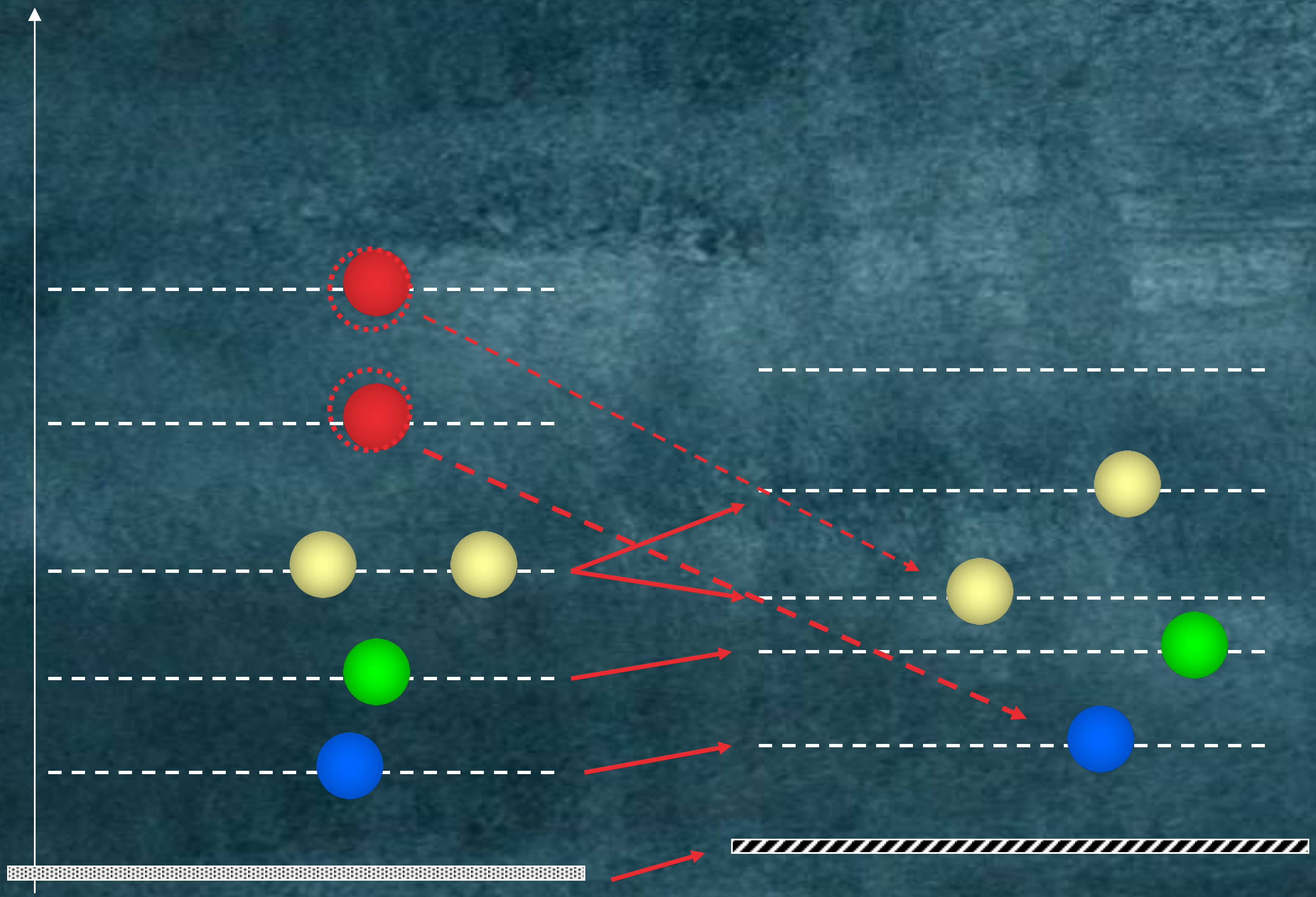
Elastic process

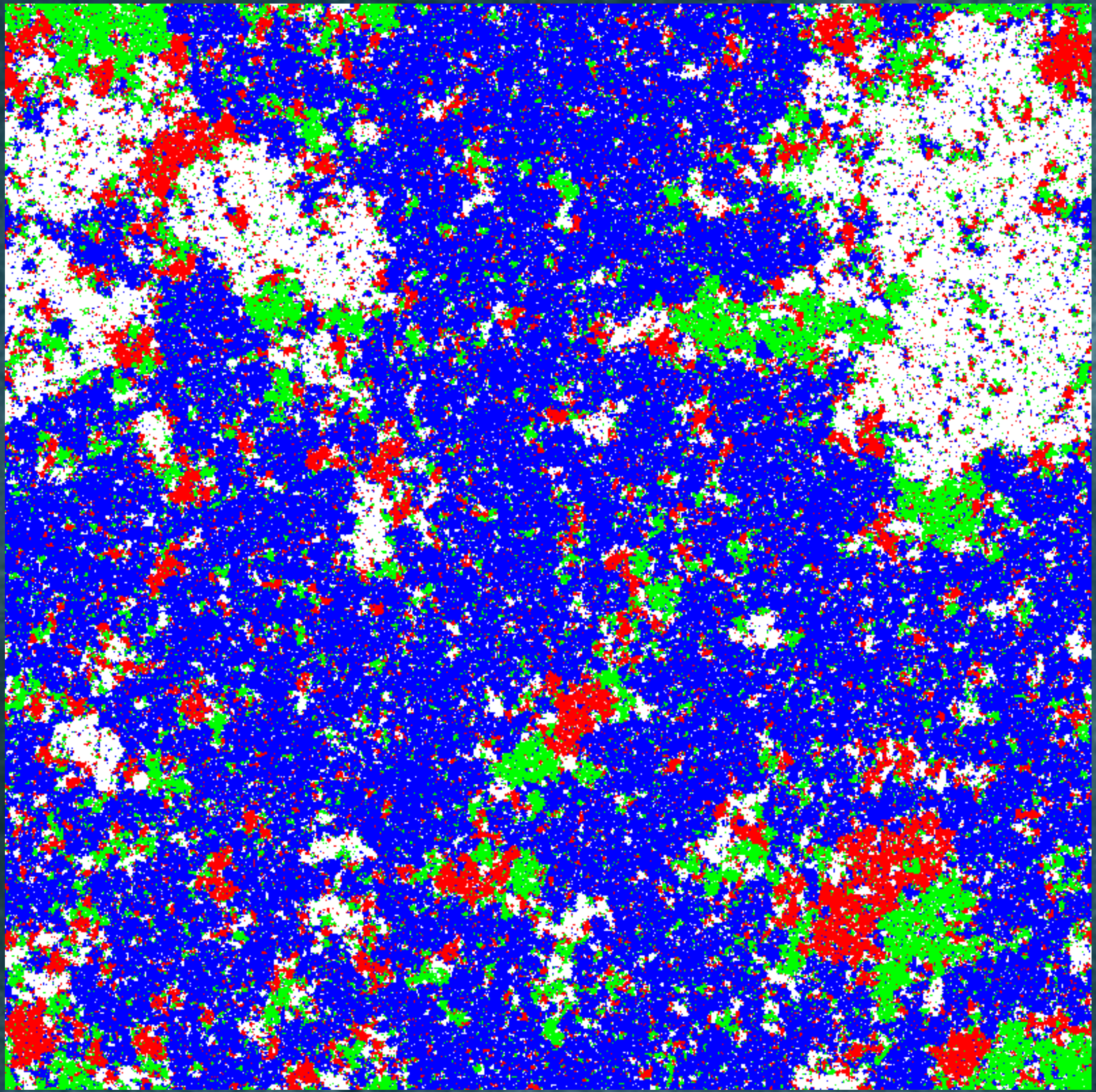


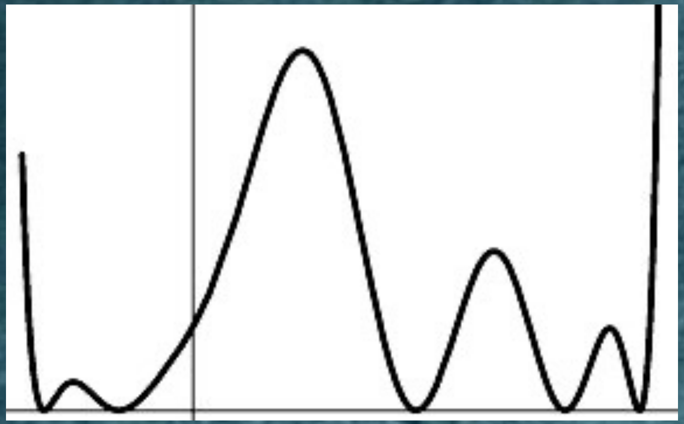
Production processes

Analytic structure of the S-matrix







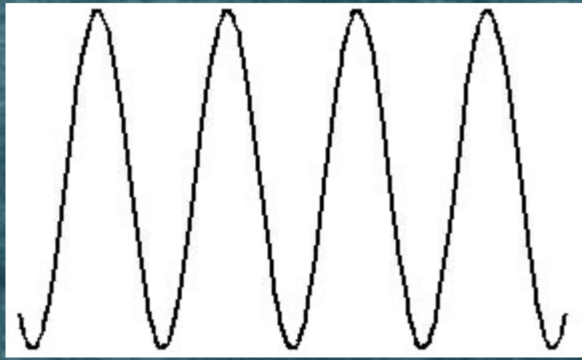


- Can we determine the number of stable scalar particles around each vacuum?
- What is the value of their mass?
- What is the decay rate of the resonances?
- Can we compute the elastic part of the S-matrix of the kinks below threshold?
- How much stable are the kinks?!

Caution

Ordinary Perturbation Theory may give wrong answer

- Sine-Gordon model



$$V(\phi) = \frac{m^2}{\beta^2}(1 - \cos\beta\phi) \simeq \frac{m^2}{2}\phi^2 + \cdots$$

It seems that there is always a scalar excitation on each vacuum

But, from the exact solution of the model, there are actually none if

$$\beta^2 \geq 4\pi$$

Although not exactly solvable, a big deal of information can be also extracted when the systems are

non-integrable

- Truncated Conformal Space Approach

  (Quite an efficient numerical method, different from Montecarlo)

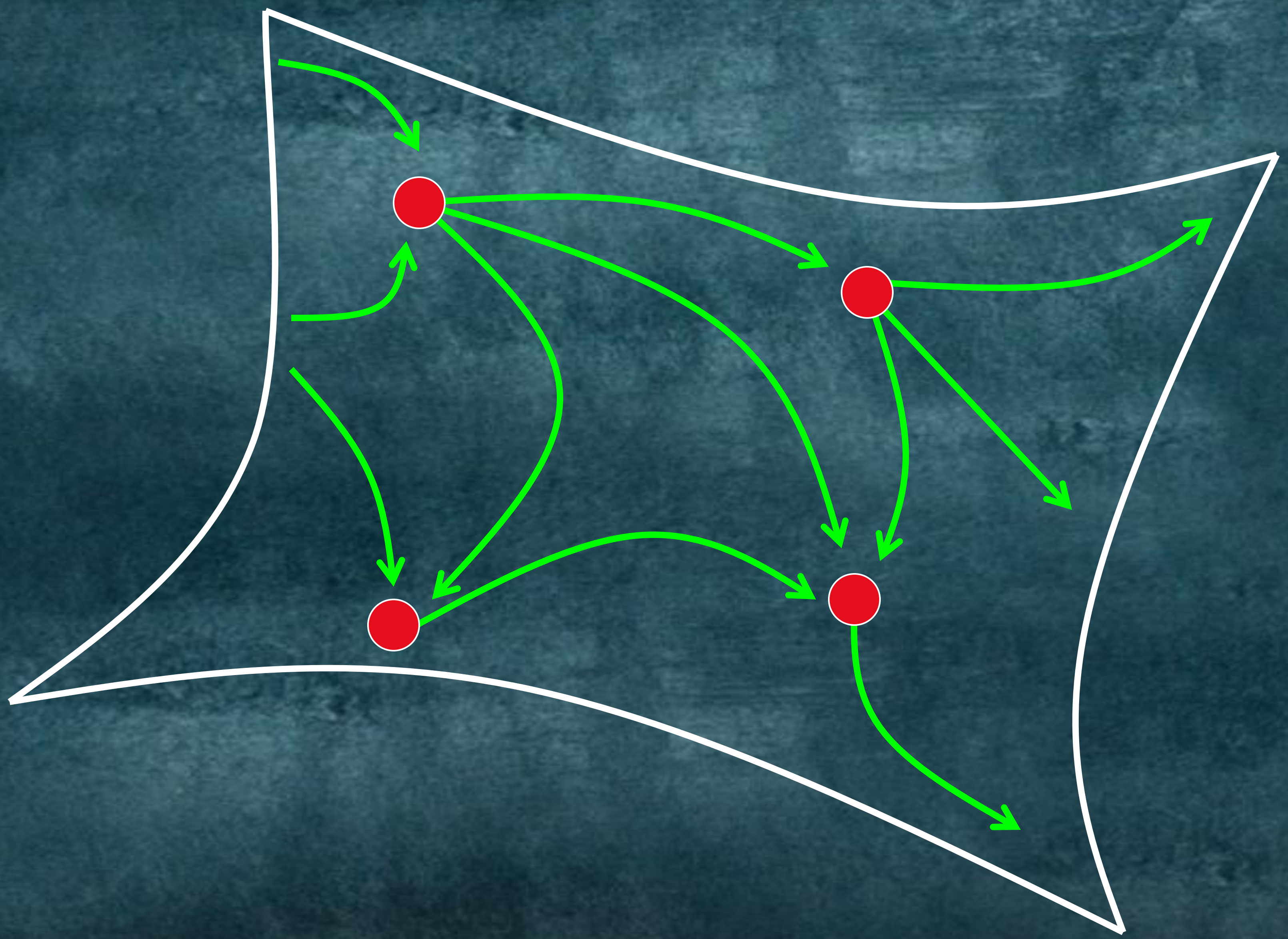
- Semi-classical method

  (It works particularly well in presence of kink excitations)

- Form-Factor Perturbation Theory

  (Well suited for studying the breaking of integrable models and the evolution of their spectrum)

QFT as RG Flows



Truncated Conformal Space Approach

(Yurov, Al. Zamolodchikov;
GM, Konik, Takacs, Rychkov…)

$$H = H_0 + \lambda \int_0^R dx\, \phi(x) =$$

$$= H_{CFT} + V$$

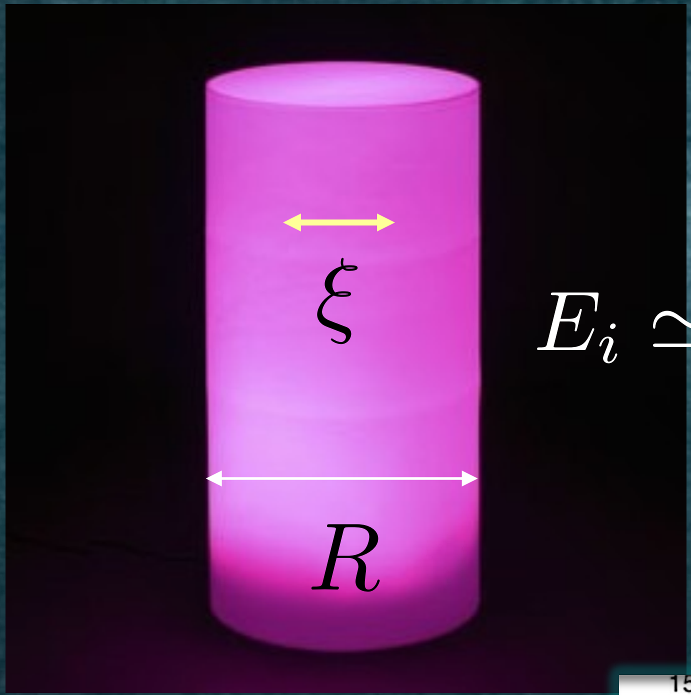
Matrix elements on the conformal states

$$< n \,|\, H_{CFT} \,|\, m > \; = \frac{2\pi}{R}\left(\Delta_n + \bar{\Delta}_n - \frac{c}{12}\right)\delta_{n,m}$$

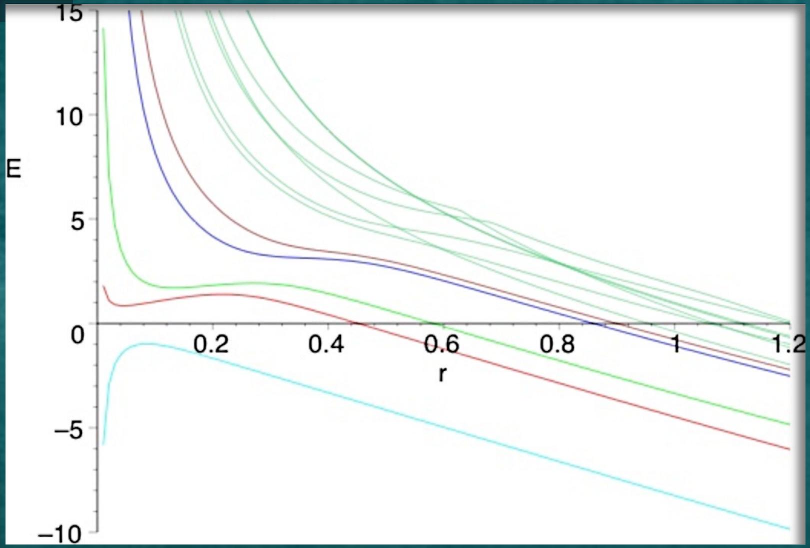
$$< n \,|\, V \,|\, m > \; = \lambda \left(\frac{2\pi}{R}\right)^{2\Delta_\varphi - 1} C^{\varphi}_{nm}$$

$$H = \frac{2\pi}{R}\begin{pmatrix} *\,00000000........ \\ 0\,*\,0000000........ \\ 00\,*\,000000........ \\ 0\,000\,*\,0000...... \\ 0\,0000\,*\,000..... \\ .... \end{pmatrix} + R^{1-2\Delta}\begin{pmatrix} ***********.... \\ ***********.... \\ ***********.... \\ ***********.... \\ ***********.... \\ .... \end{pmatrix}$$

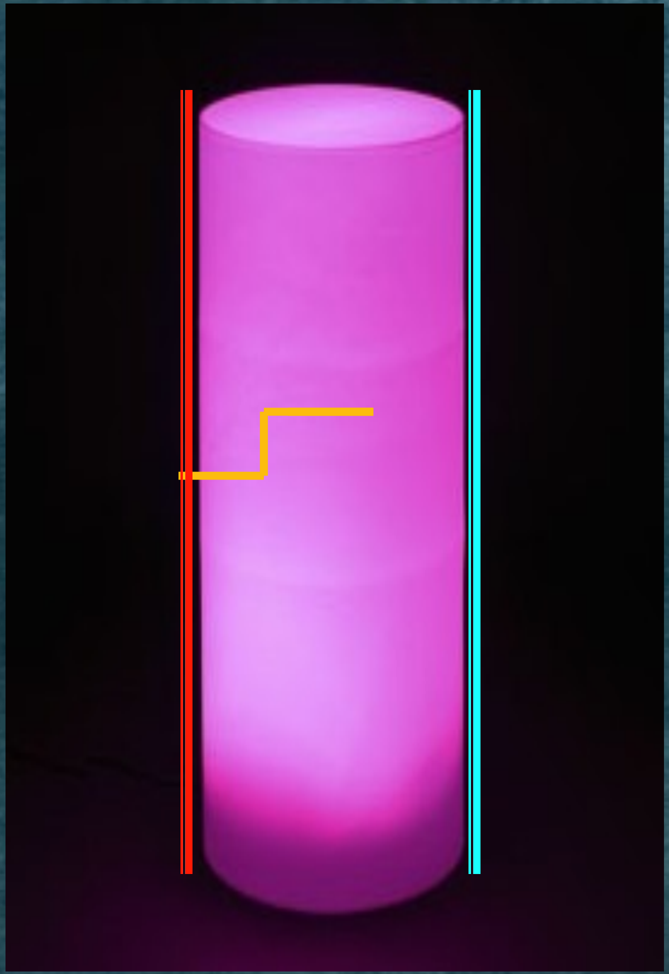
Behavior of the eigenvalues



$$E_i \simeq \begin{cases} \frac{2\pi}{R}\left(\Delta_i + \overline{\Delta}_i - \frac{c}{12}\right) & , \quad R \ll \xi \\ \frac{\epsilon_0}{\xi^2} R + \sum_i m_i & , \quad R \gg \xi \end{cases}$$

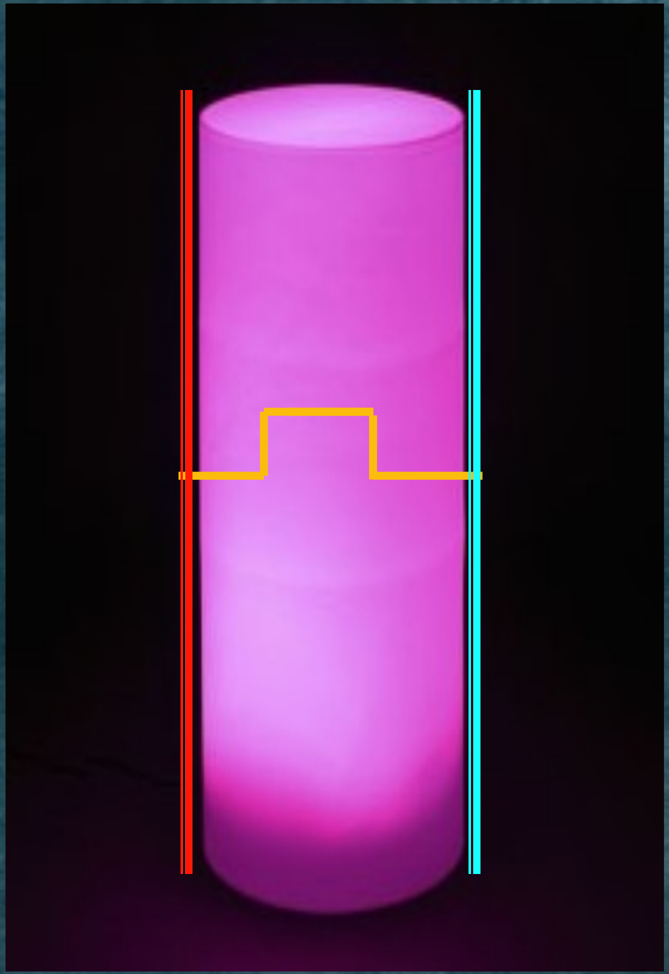


Role of the boundary conditions



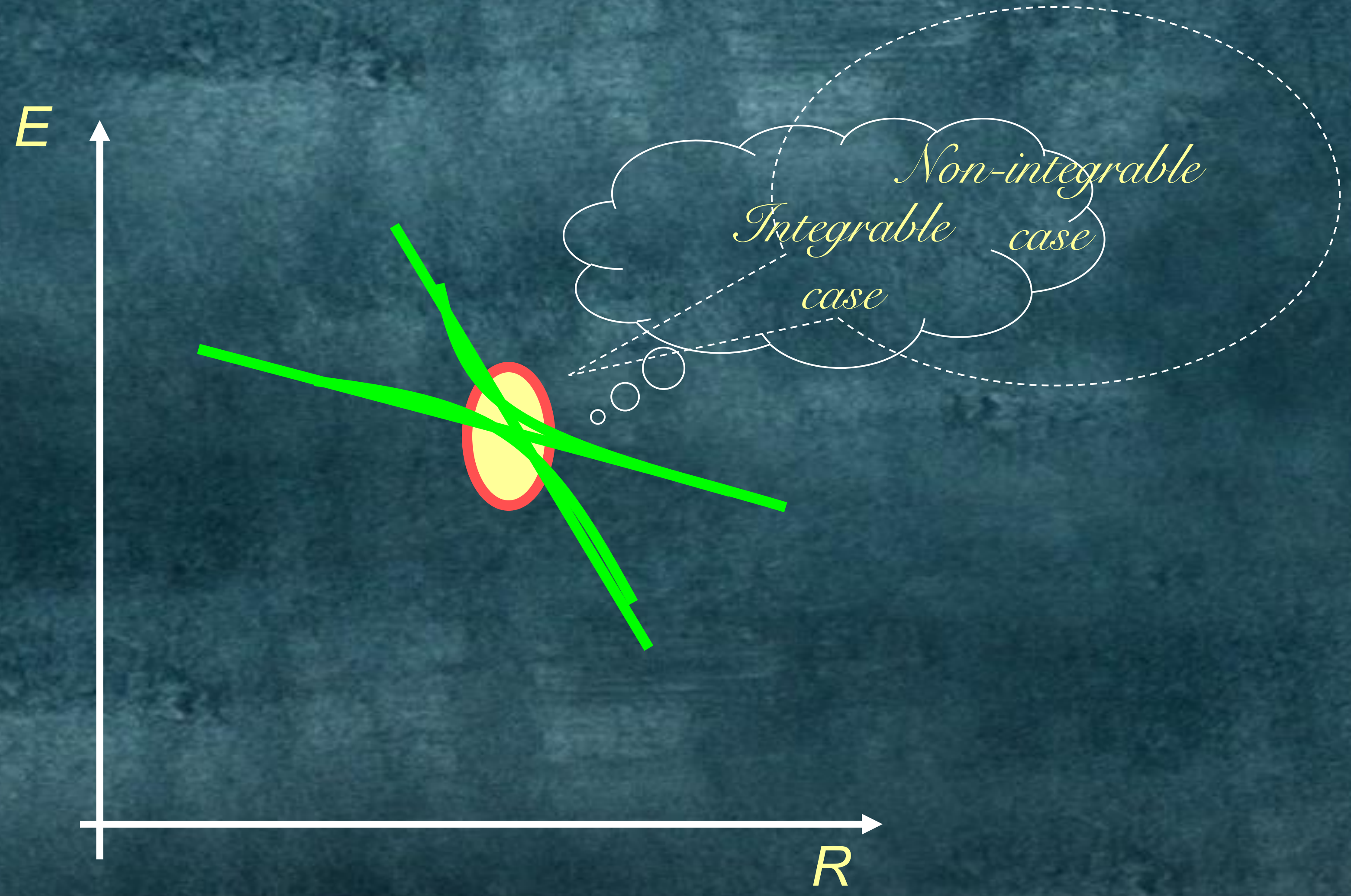
- Kink state exist only with anti-periodic or twisted b.c.

Role of the boundary conditions

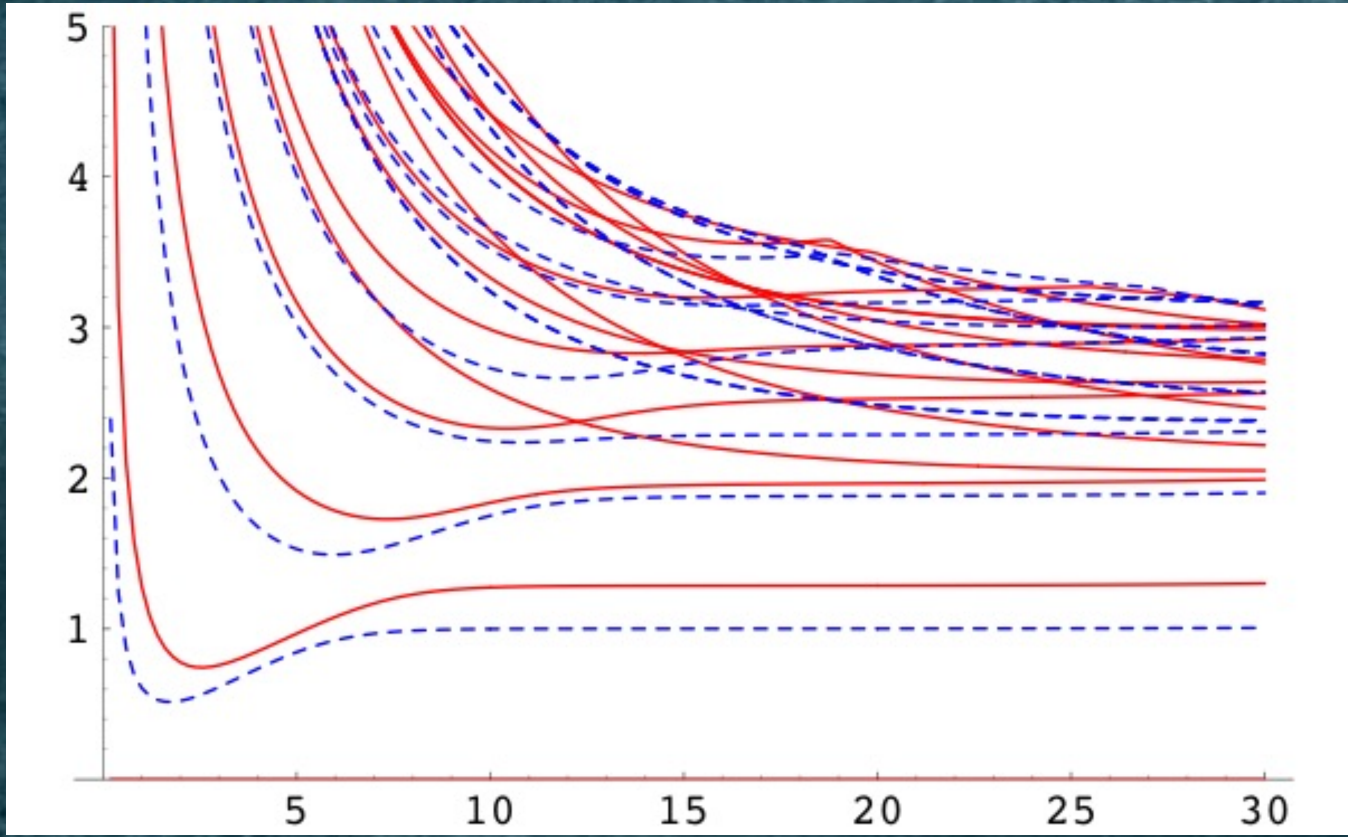


- Only kink-antikink states are present with periodic b.c.

Integrable –non integrable signatures

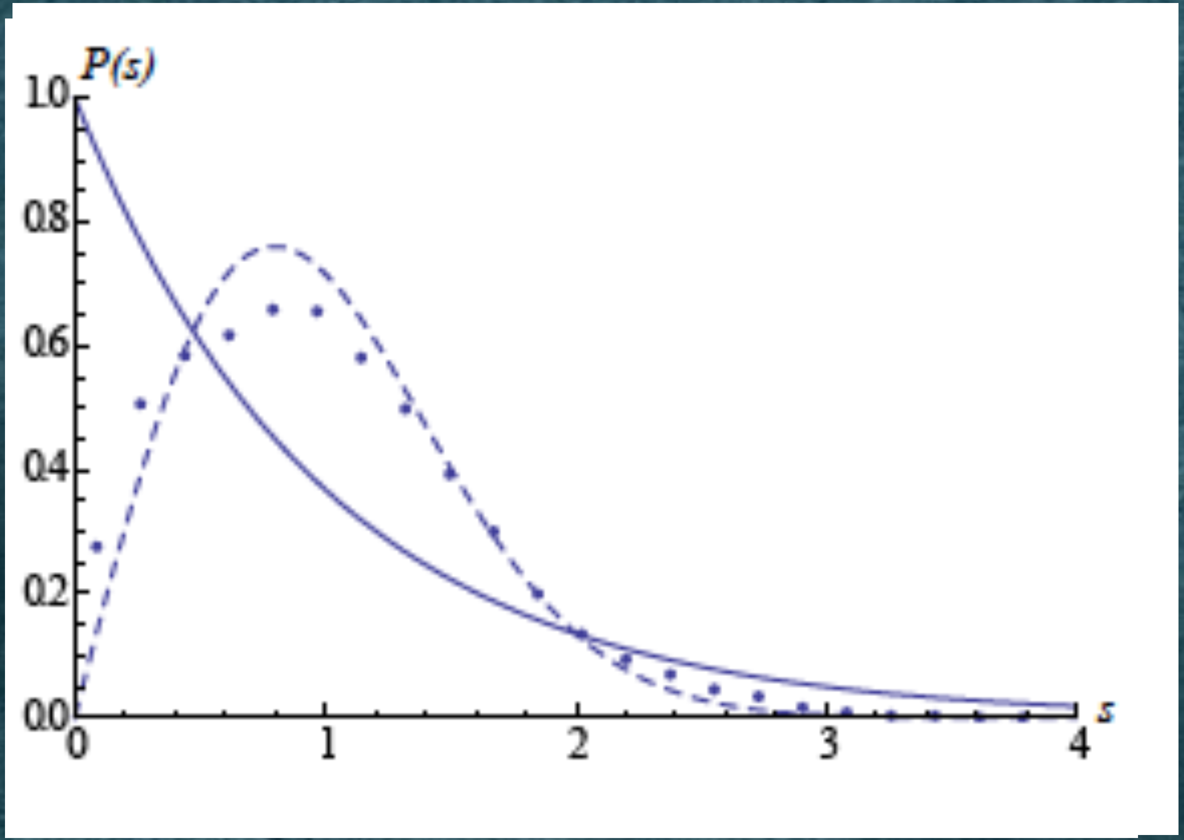


Integrable vs non-integrable



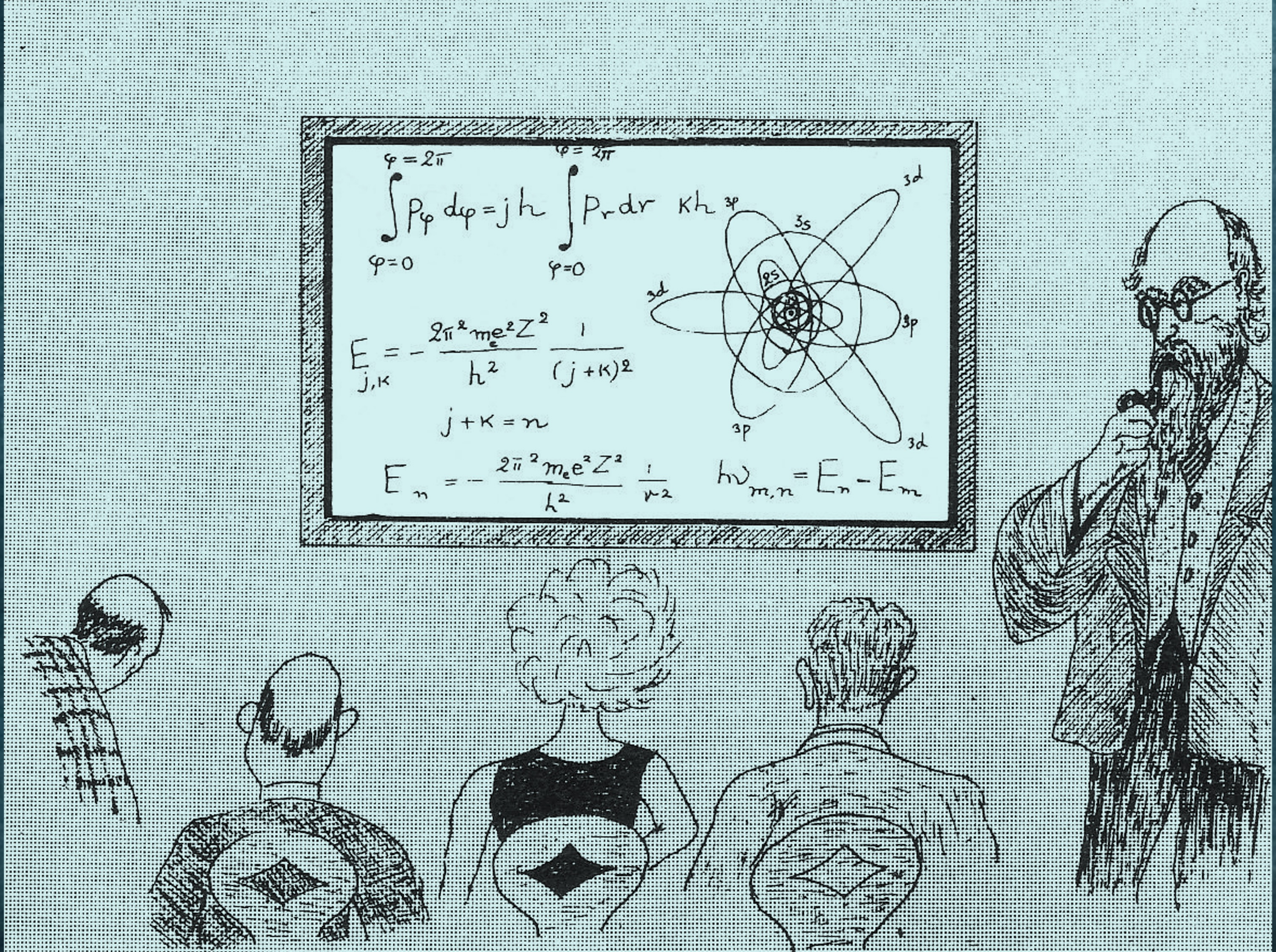
Such aspects have been analysed in deformed CFT

(Brandino, Konik, GM)



$$H = H_{CFT} + (T - T_c)\int \epsilon(x)\, dx$$

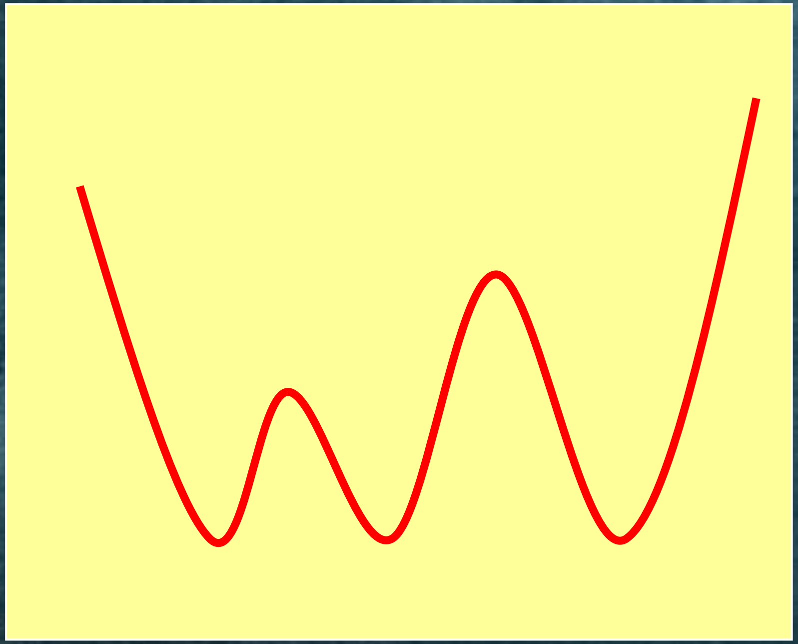
$$H = H_{CFT} + h \int \sigma(x)\, dx$$


$\int_{\varphi=0}^{\varphi=2\pi} p_\varphi \, d\varphi = jh \quad \int_{\varphi=0}^{\varphi=2\pi} p_r \, dr \quad \kappa h$
$E_{j,\kappa} = -\frac{2\pi^2 m_e e^2 Z^2}{h^2} \frac{1}{(j+\kappa)^2}$
$j + \kappa = n$
$E_n = -\frac{2\pi^2 m_e e^2 Z^2}{h^2} \frac{1}{n^2}$
$h\nu_{m,n} = E_n - E_m$
3d
3p
3s
2s
3d
3p
3p
3d

Semi-classical methods

$$L = \frac{1}{2}\left(\partial_\mu \Phi\right)^2 - U(\Phi)$$

Where U(Φ) has a set of degenerate vacua, e.g.



Semi-classical methods

$$L = \frac{1}{2}\left(\partial_\mu \Phi\right)^2 - U(\Phi)$$

Where U(Φ) has a set of degenerate vacua, e.g.

(i) Sine-Gordon

$$U(\varphi) = \frac{m^2}{\beta^2}\left(1 - \cos\beta\varphi\right)$$

(ii) Broken phase of Φ^4 theory

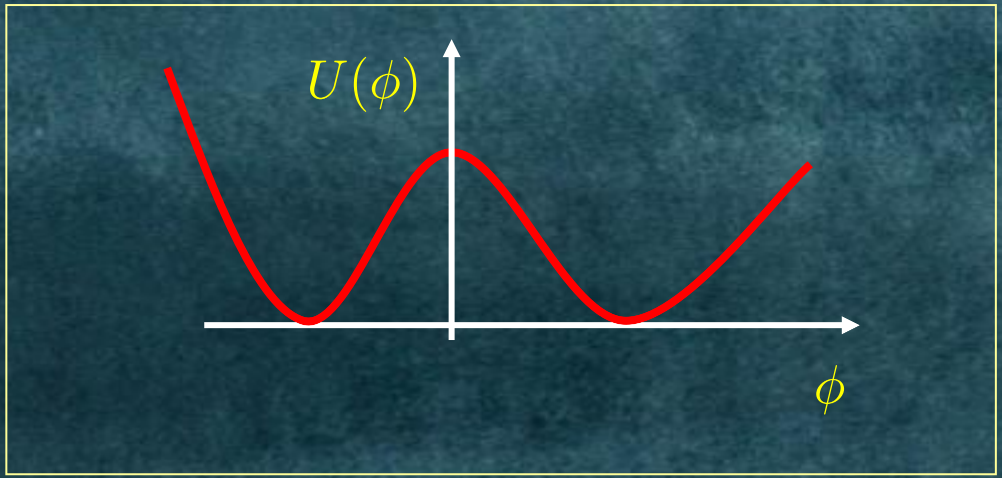
$$U(\varphi) = \frac{\lambda}{4}\left(\varphi^2 - \frac{\mu^2}{\lambda^2}\right)^2$$

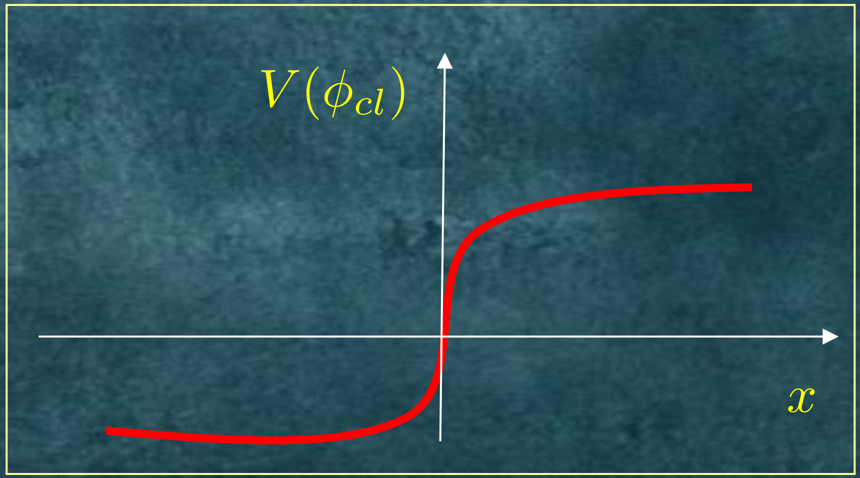
(iii) Multi-frequency Sine-Gordon

$$U(\varphi) = \mu\,\cos\beta\varphi + \lambda\,\cos\omega\varphi$$

The method works equally well also in presence of fermions

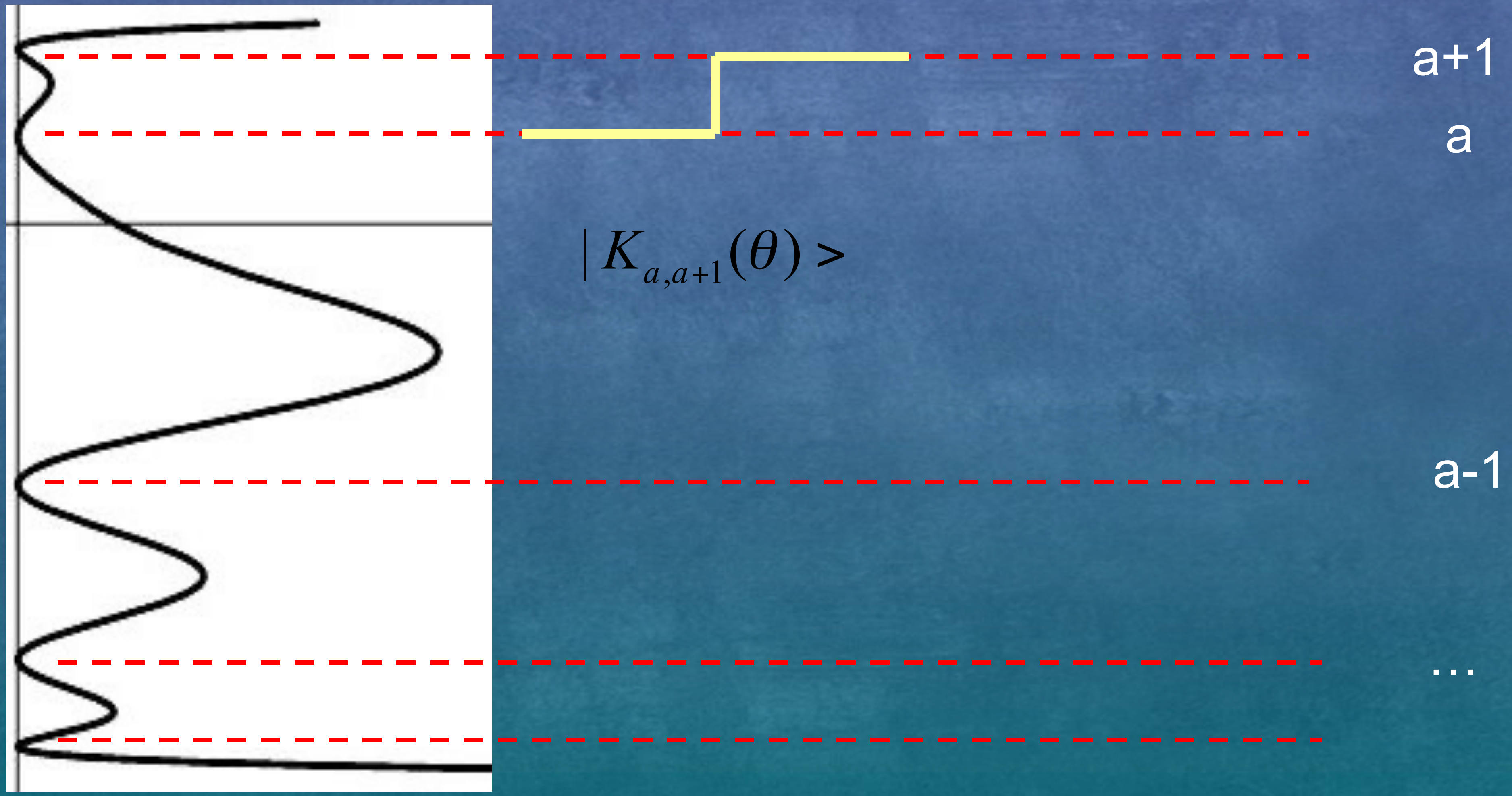
$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - U(\phi) + i\bar{\psi}\gamma^\mu\partial_\mu\psi - gV(\phi)\bar{\psi}\psi$$





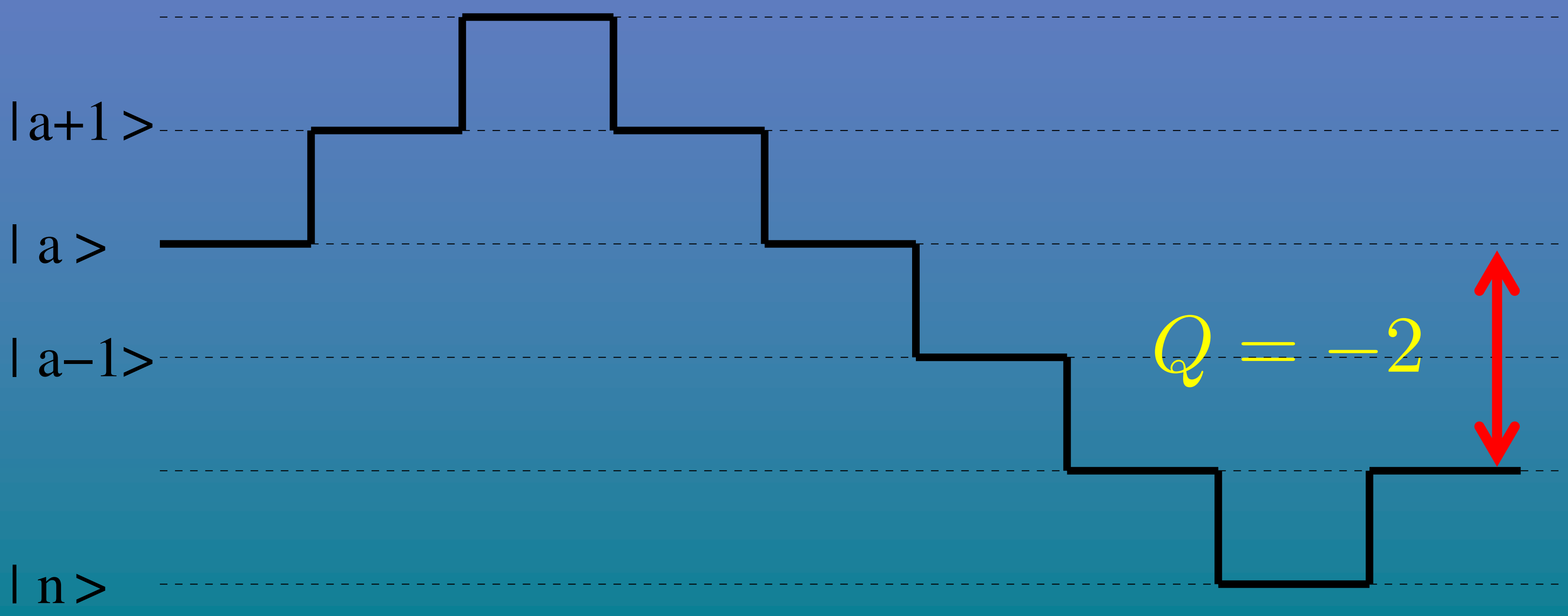
SUSY: $U(\phi) = [W'(\phi)]^2$ $V(\phi) = W''(\phi)$

Basic excitations: kinks and anti-kinks



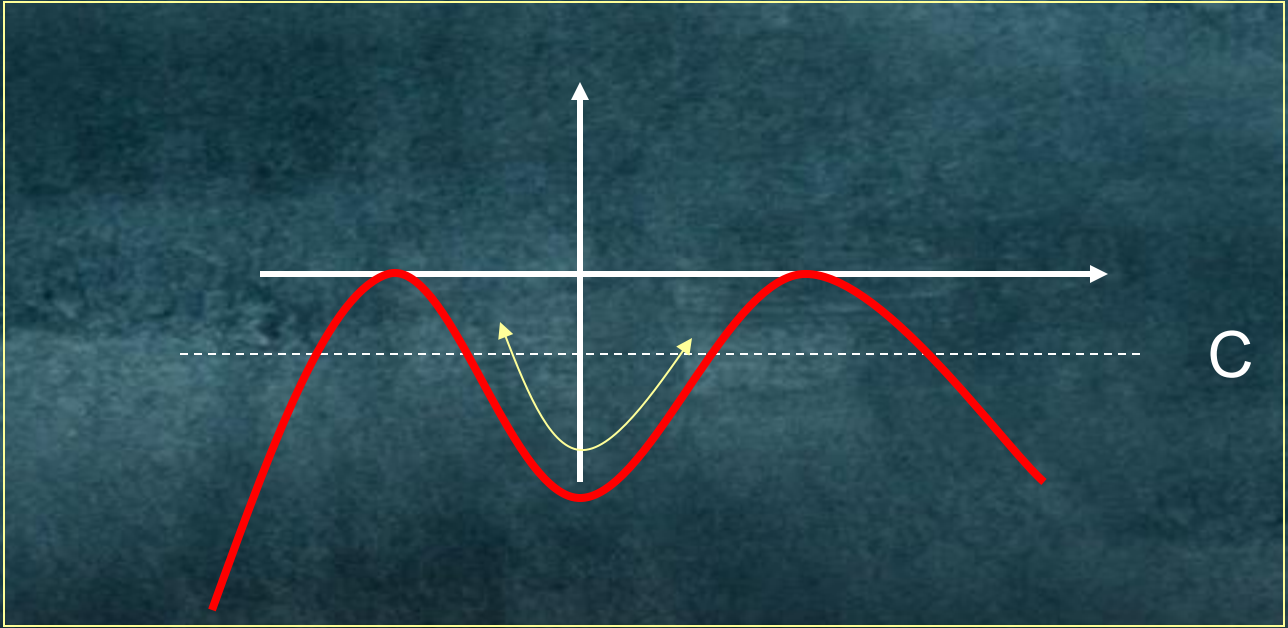
Q-topological sector

$$Q = \varphi(\infty) - \varphi(-\infty)$$

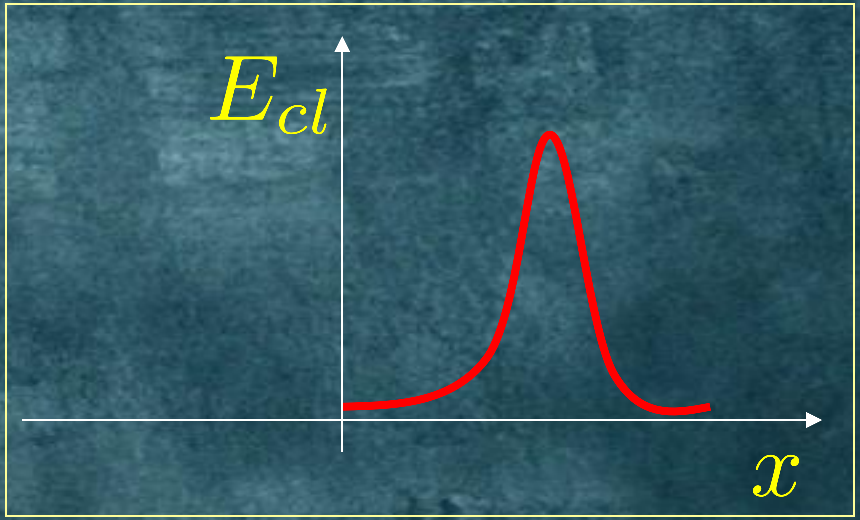


The classical kink configurations are obtained by integrating

$$\frac{1}{2}\left(\frac{\partial \Phi_{cl}}{\partial x}\right)^2 = V(\Phi_{cl}) + C$$



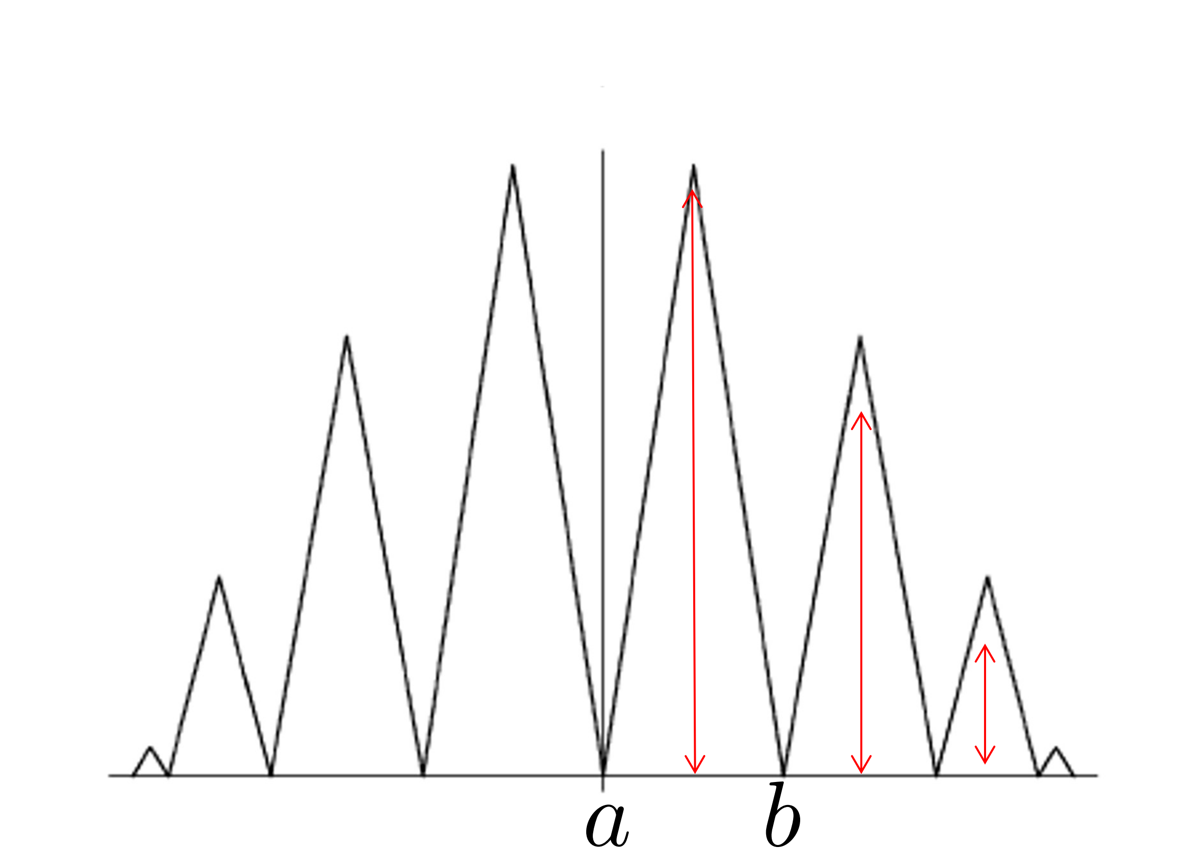
Semi-classical data



$$M_{cl} = \begin{cases} \dfrac{8m}{\beta^2} & SG \\ \dfrac{2\sqrt{2}}{3}\,\dfrac{m^3}{\lambda} & \Phi^4 \end{cases}$$

Mass of the kinks: rule of thumb and BPS

$$M_{ab} = \int_{\varphi_a}^{\varphi_b} \sqrt{2U(\varphi)}\, d\varphi \quad = Z_{ab}$$



Bosonic Kink Form Factor

(Goldstone-Jackiw, GM)

The semi-classical two-particle matrix element of the fundamental field among kink states is simply the Fourier transform of the classical solution

$$\langle K^{\pm}_{ab}(\theta_1) \mid \varphi(0) \mid K^{\pm}_{ab}(\theta_2)\rangle \equiv F(\theta)$$

$$\theta = \theta_1 - \theta_2$$

$$F(\theta) = \int da \; e^{iM_{cl}\theta\, a} \Phi_{cl}(a)$$

Its proof is simple

- Heisenberg equation of motion

$$\left(\partial_t^2 - \partial_x^2\right)\,\Phi(x,t) \,=\, -V'[\Phi(x,t)]$$

Its proof is simple

- Heisenberg equation of motion

$$\left(\partial_t^2 - \partial_x^2\right) \langle K(\theta_1)|\Phi(x,t)|K(\theta_2)\rangle = -\langle K(\theta_1)|V'[\Phi(x,t)]|K(\theta_2)\rangle$$

- Extract the (x,t) dependence and use the rapidity difference to define

$$f(a) = \int \frac{d\theta}{2\pi} e^{-iM_{cl}\theta a} F(a)$$

$$F(\theta) \equiv \langle K(\theta_1)|\Phi(0)|K(\theta_2)\rangle$$

$$\theta = \theta_1 - \theta_2$$

- Left hand side

$$2\,M_{cl}^2\,(1-\cosh\theta)\;\simeq\;-(M_{cl}\theta)^2\,F(\theta)$$

- Right hand side (semi-classically) is saturated by the lowest intermediate states

- Hence f(a) satisfies the same differential equation satisfied by the classical kink solution

$$\frac{d^2}{da^2}f(a)=V^{'}[f(a)]$$

Semi-classical Form Factor of other bosonic operators

$$F^{G}(\theta) = \int da\, e^{iM\theta a}\, G[\Phi_{cl}(a)]$$

Example

$$F^{\Phi^2}(\theta) = \int da\, e^{iM\theta a}\, \Phi_{cl}^2(a)$$

Bound states

(GM)

The semi-classical expression of the Form Factors permits to easily obtain the bound states of the theory

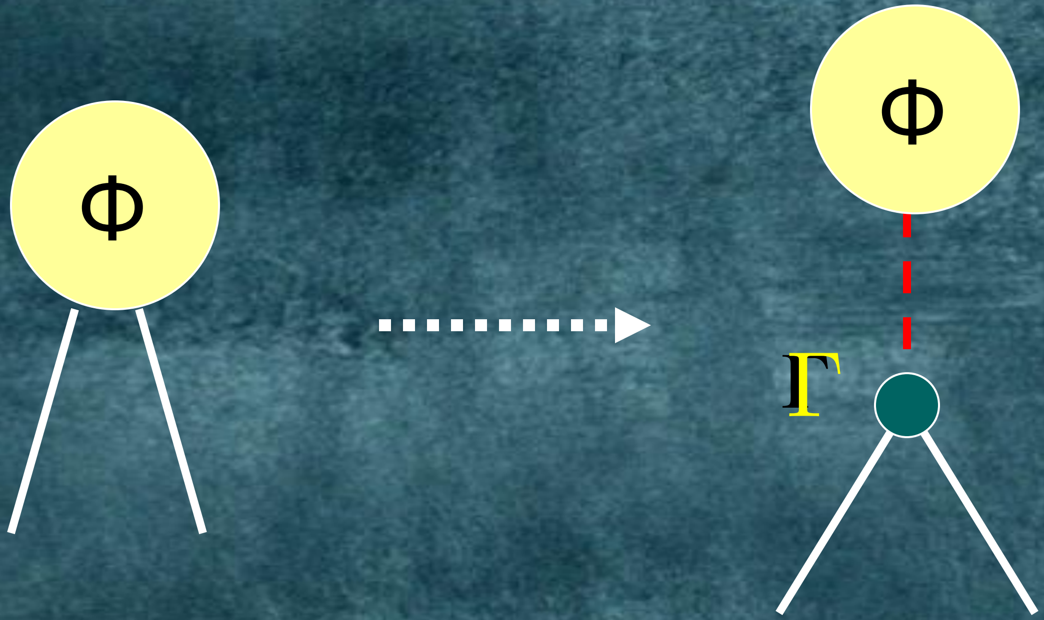
Bound states

(GM)

To this aim, consider the poles of the crossed channel FF

$$\theta \to i\pi - \theta$$

$$\hat{F}(\theta) \equiv \langle a \mid \varphi(0) \mid K^{\pm}_{ab}(\theta_1) K^{\pm}_{ba}(\theta_2) \rangle$$

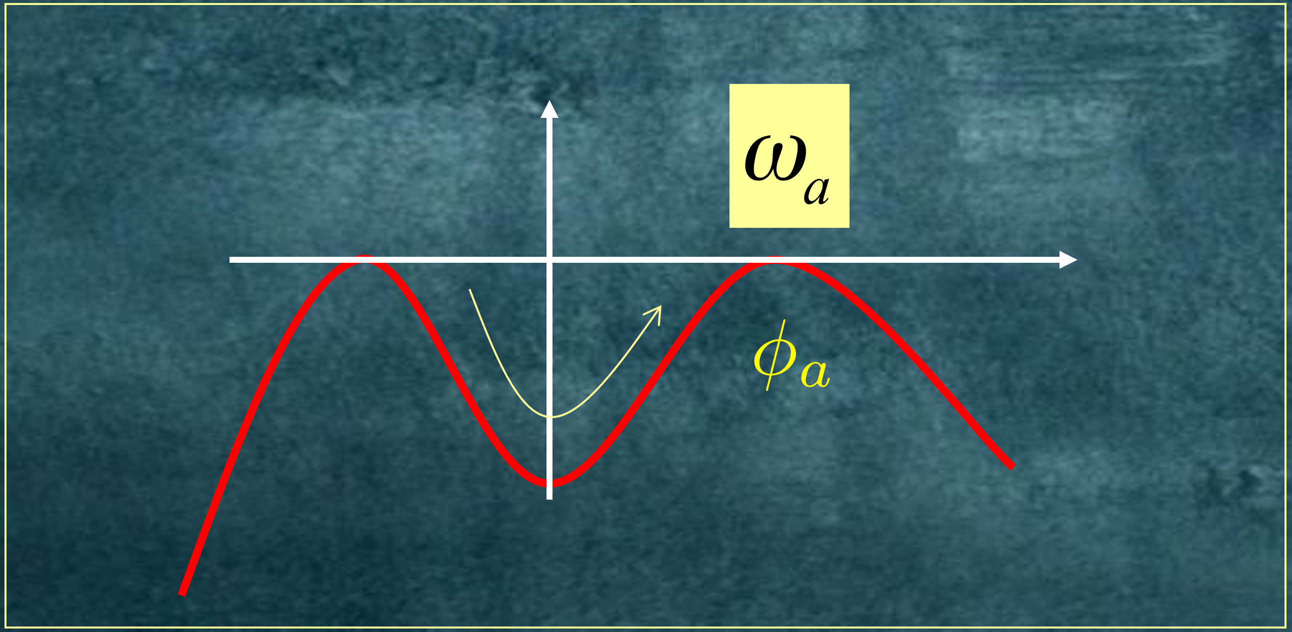


$$G(\theta) \simeq i \, \frac{\Gamma^{b_n}_{K\bar{K}}}{(\theta - i u_n)} \langle 0|\phi(0)|b_n\rangle$$

If the resonance angle is within the physical strip, the masses of the bound states are

$$m_n = 2 M_{cl} \sin \frac{u_n}{2}$$

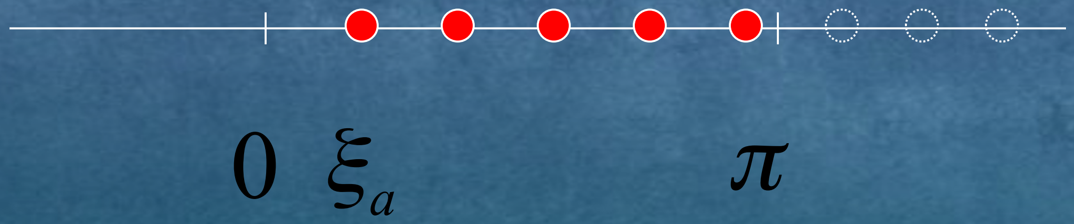
Structure of the poles



$$\xi_a = \frac{\omega_a}{M}$$

$$\phi(x) = \phi_a + \sum_n b_n \exp\left(-n\,\xi_a x\right)$$

Poles in its Fourier transform



Most general expression of semiclassical Form Factors

$$\varphi(x) \approx \sum_n b_n \exp(-n\xi_a x)$$

$$\Downarrow$$

$$F^G(k) = \int da\, e^{ika} G[\varphi(a)] = \sum_n \frac{g_n}{ik + n\xi}$$

The poles are fixed. The information on the operator G enters only in the coefficients g_n

Universal Mass Formulas

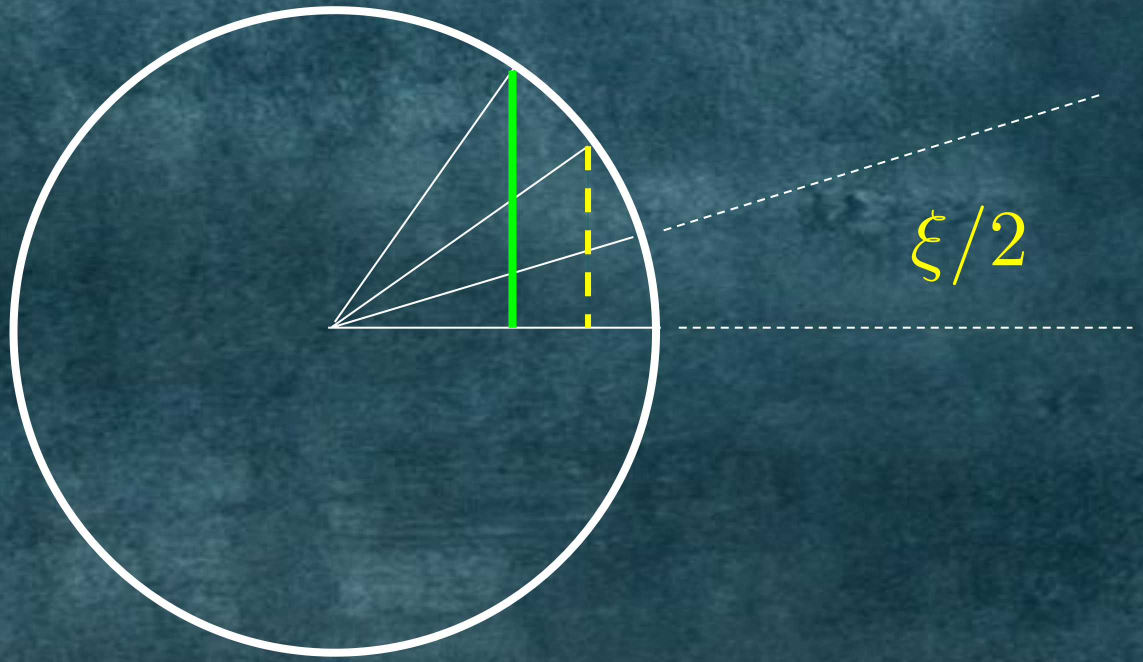
Mass of the bound states

$m_a^B(n) \;=\; 2M\,\sin\frac{n\,\pi\,\xi_a}{2}$	$n = 1, 2, \ldots \left[\frac{1}{\xi_a}\right]$

However, not all of them are necessarily stable!

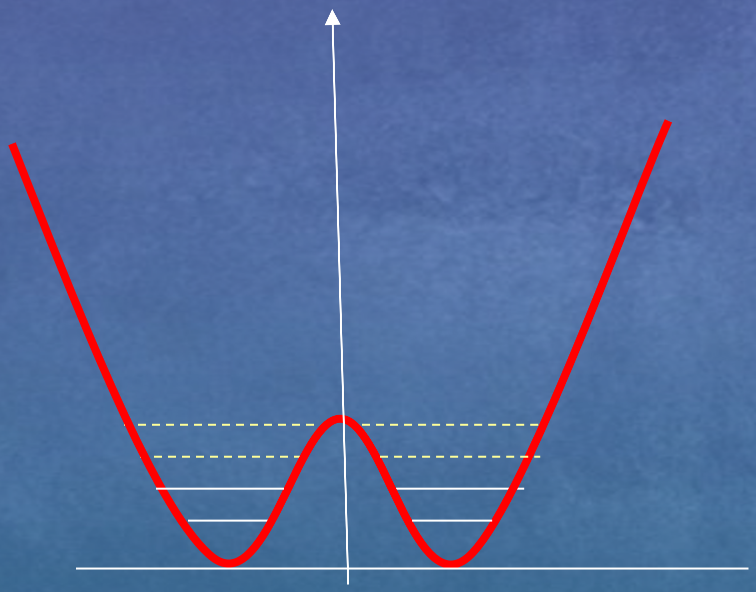
$$m_k > 2m_1$$

$$k = 3, 4, 5, \ldots$$



Conclusion: in a non-integrable QFT the number of stable particles around each vacuum can be at most 2

Φ^4 in its broken phase



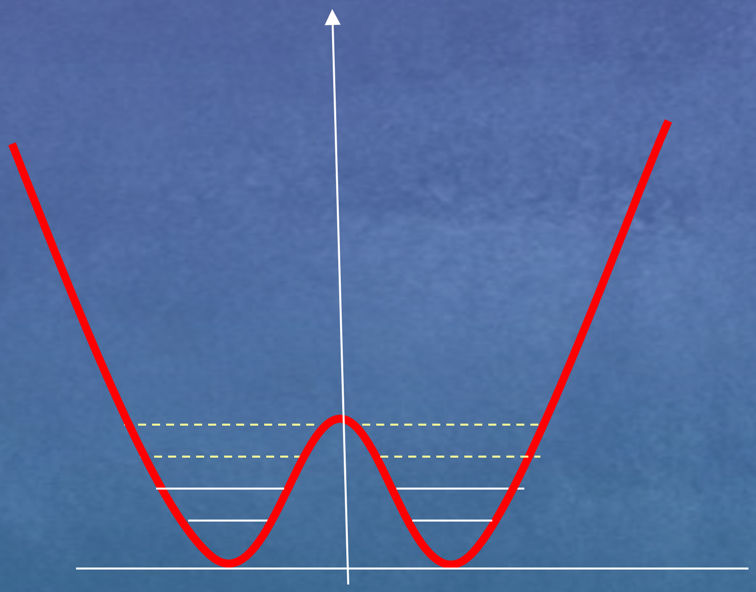
$$U(\Phi) = \frac{\lambda}{4}\left(\Phi^2 - \frac{\mu^2}{\lambda}\right)^2$$

$$\xi = \frac{3\lambda}{2\pi\mu^2}$$

$$< a \,|\, \Phi(0) \,|\, K_{-a,a}(\theta_1) K_{a,-a}(\theta_2) > \; = \; \frac{1}{\sinh \frac{i\pi - \theta}{\xi}}$$

If $\xi > 1$, i.e. $\frac{\lambda}{\mu^2} > \frac{2\pi}{3}$ no bound states

Φ^4 in its broken phase



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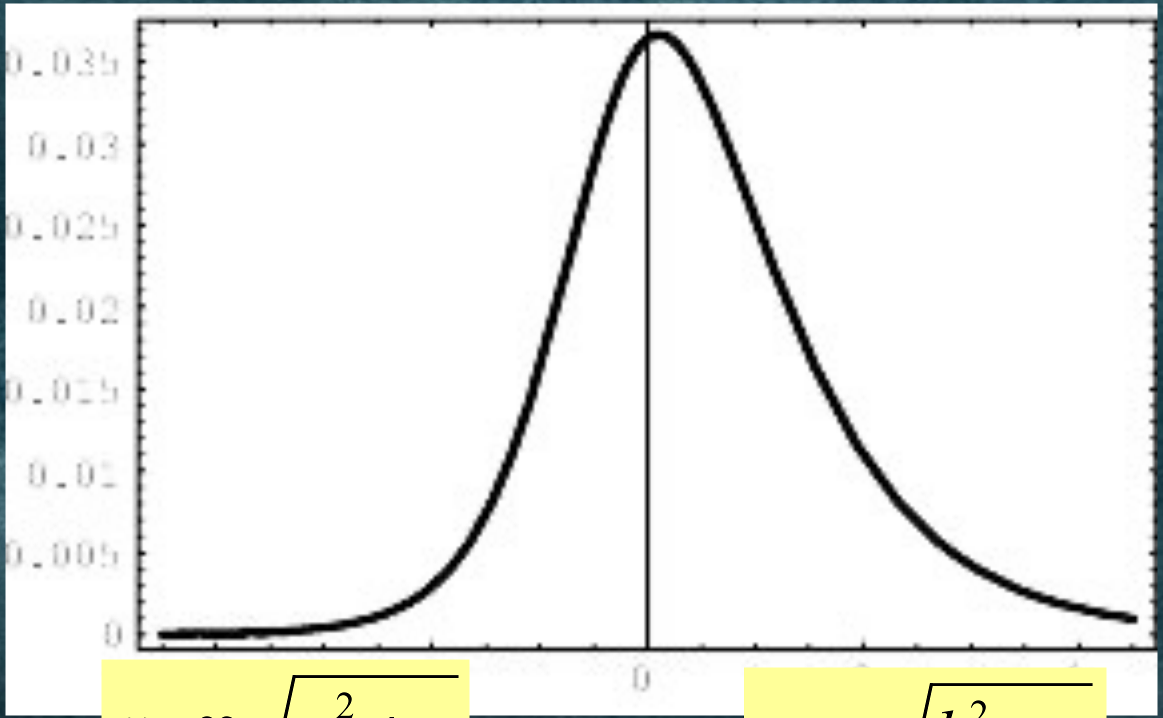
$$\xi = \frac{3\lambda}{2\pi\mu^2}$$

$$< a \,|\, \Phi(0) \,|\, K_{-a,a}(\theta_1) K_{a,-a}(\theta_2) > \; = \; \frac{1}{\sinh \dfrac{i\pi - \theta}{\xi}}$$

If $\xi < 1$ there are instead $\left[\frac{1}{\xi}\right]$ bound states, only the first 2 stable

Asymmetric wells

$$U(\varphi) = \frac{\lambda}{2}(\varphi + a)^2 (\varphi - b)^2 (\varphi^2 + c)$$

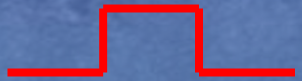


$$\omega_0 \propto \sqrt{a^2 + c}$$

$$\omega_1 \propto \sqrt{b^2 + c}$$

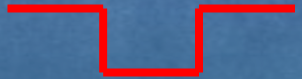
Poles and bound states

$| K(\theta_1)\overline{K}(\theta_2) >$



$$\theta = i\,\frac{\omega_0}{M}$$

$| \overline{K}(\theta_1)K(\theta_2) >$



$$\theta = i\,\frac{\omega_1}{M}$$

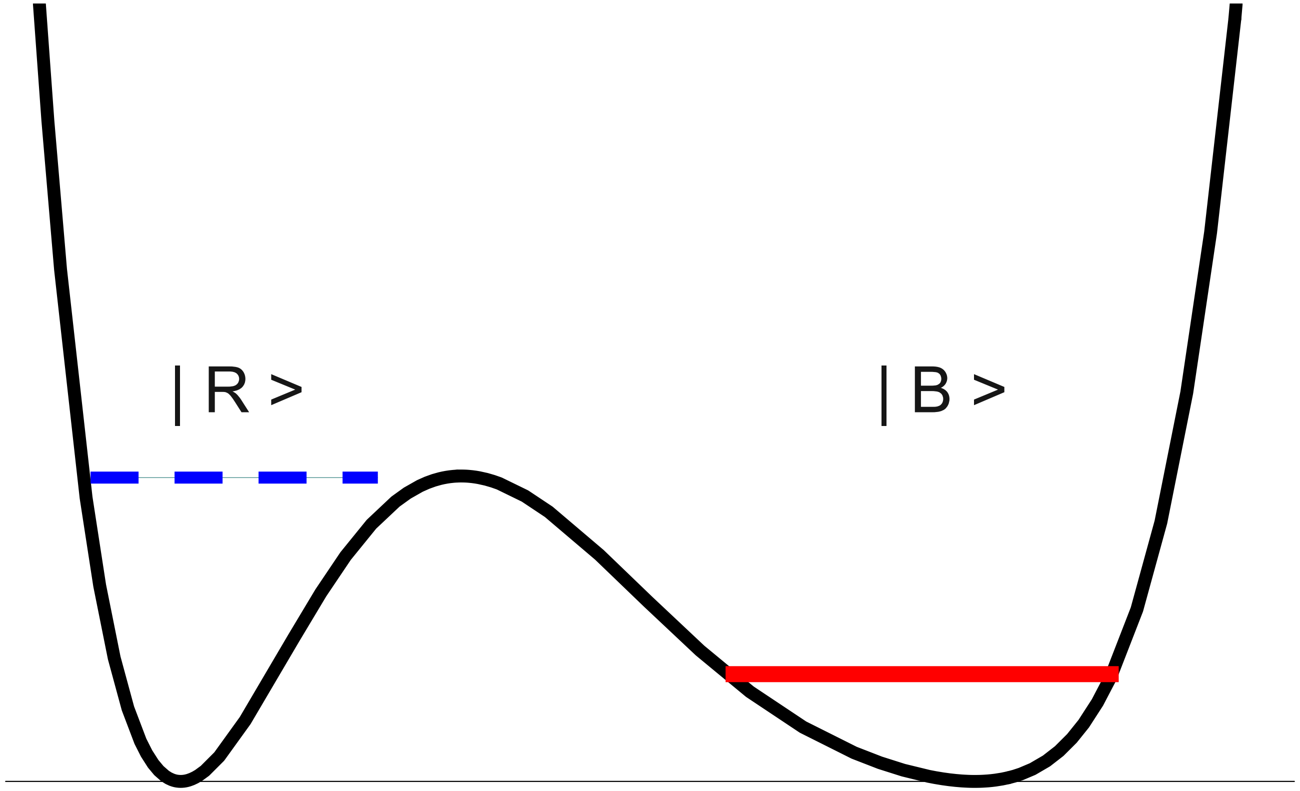
Mass of the kink

$$M = \frac{\lambda^2}{m^4}$$

If

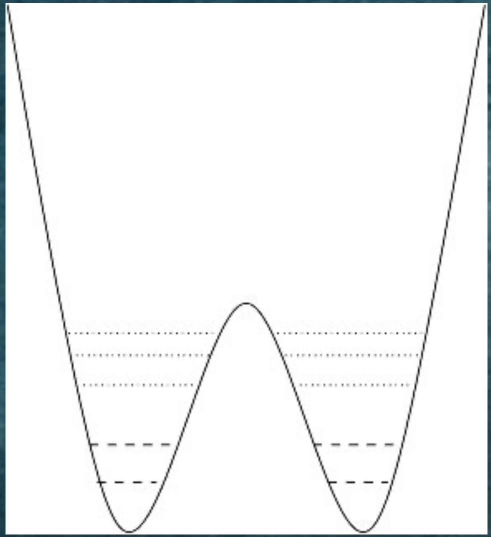
$$\frac{1}{\omega_0} < \frac{\lambda^2}{m^4} < \frac{1}{\omega_1}$$

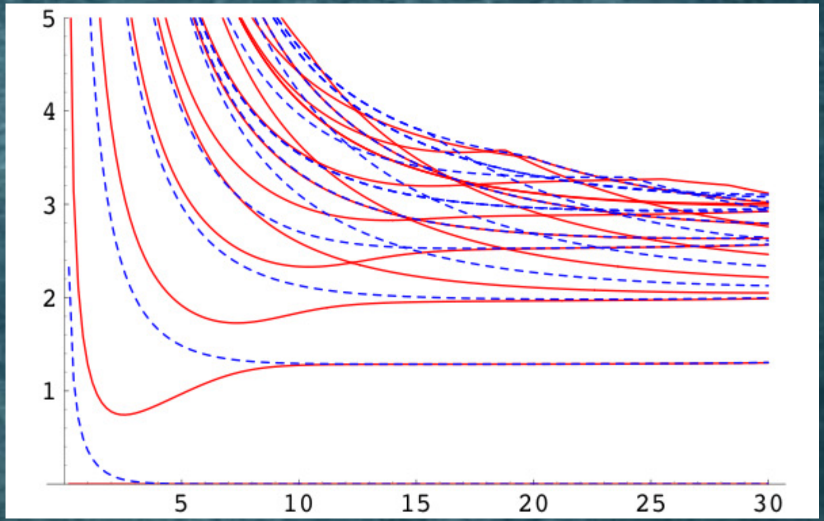
the first pole is out the physical strip but the second is inside!

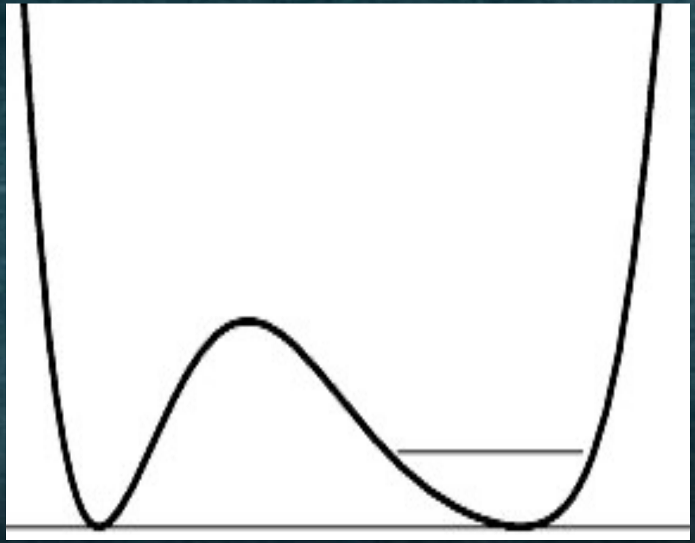

| R >
| B >

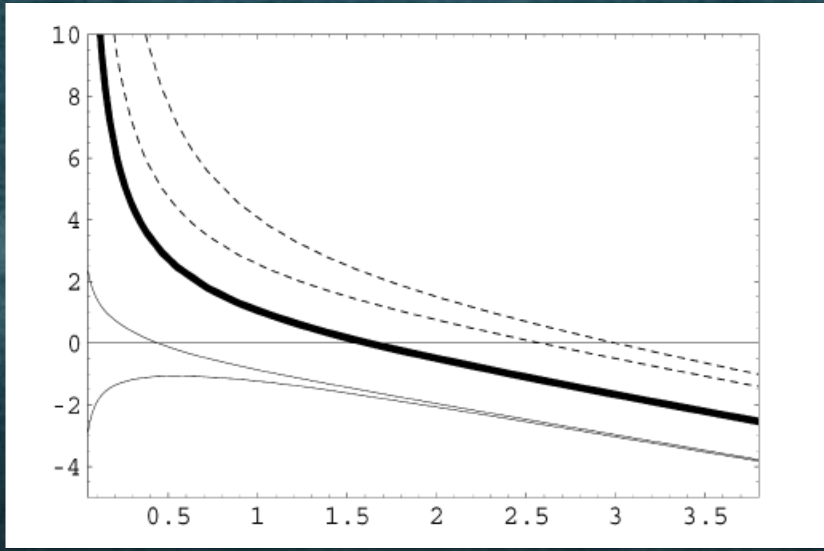
Truncated Conformal Space Approach

Lassig, GM, Cardy;
Colomo, GM, Coser…









Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

Truncated conformal space approach for 2D Landau–Ginzburg theories

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Abstract. We study the spectrum of Landau–Ginzburg theories in 1 + 1 dimensions using the truncated conformal space approach employing a compactified boson. We study these theories both in their broken and unbroken phases. We first demonstrate that we can reproduce the expected spectrum of a Φ^2 theory (i.e. a free massive boson) in this framework. We then turn to Φ^4 in its unbroken phase and compare our numerical results with the predictions of two-loop perturbation theory, finding excellent agreement. We then analyze the broken phase of Φ^4 where kink excitations together with their bound states are present. We confirm the semiclassical predictions for this model on the number of stable kink-antikink bound states. We also test the semiclassics in the double well phase of Φ^6 Landau–Ginzburg theory, again finding agreement.

Keywords: conformal field theory (theory), other numerical approaches, quantum phase transitions (theory)

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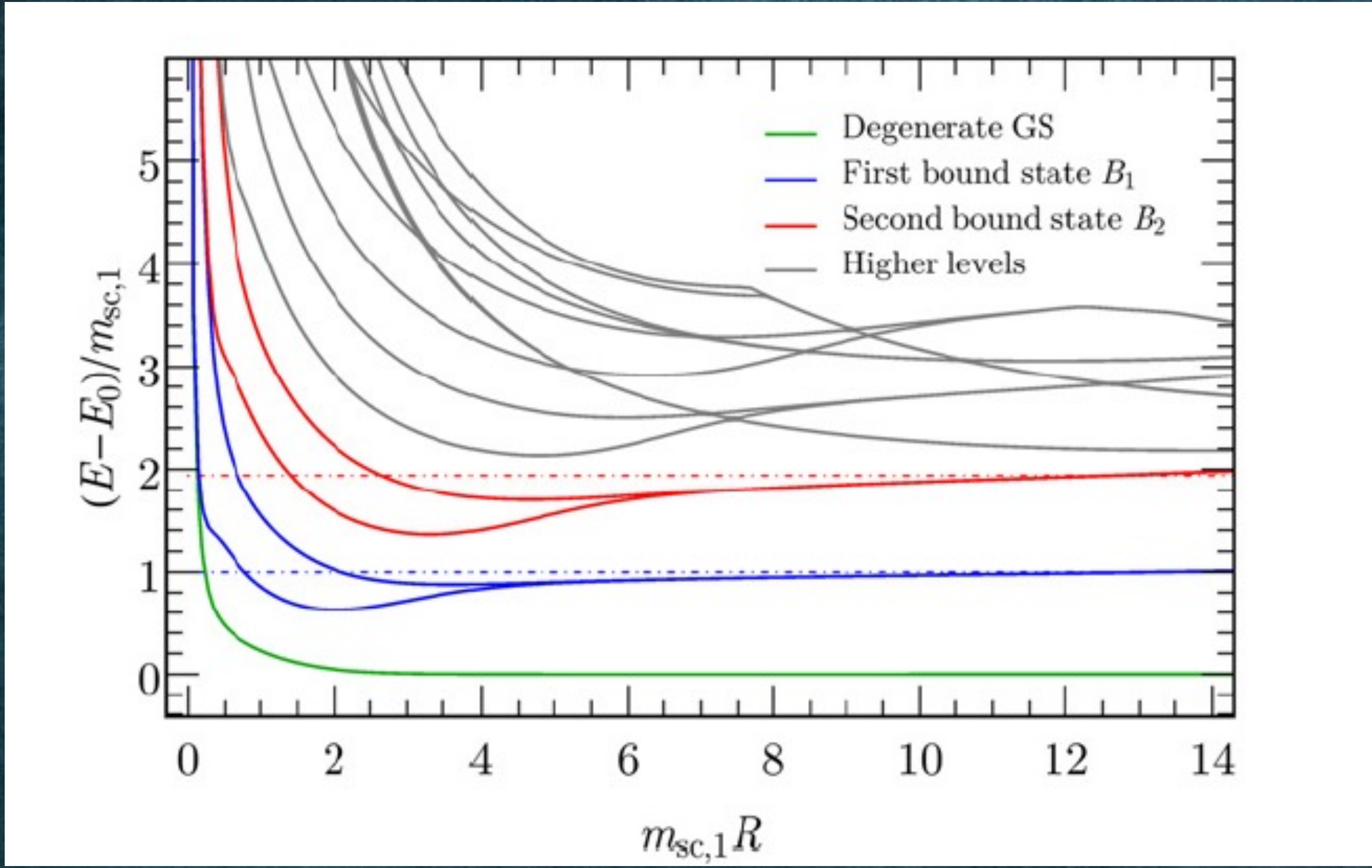
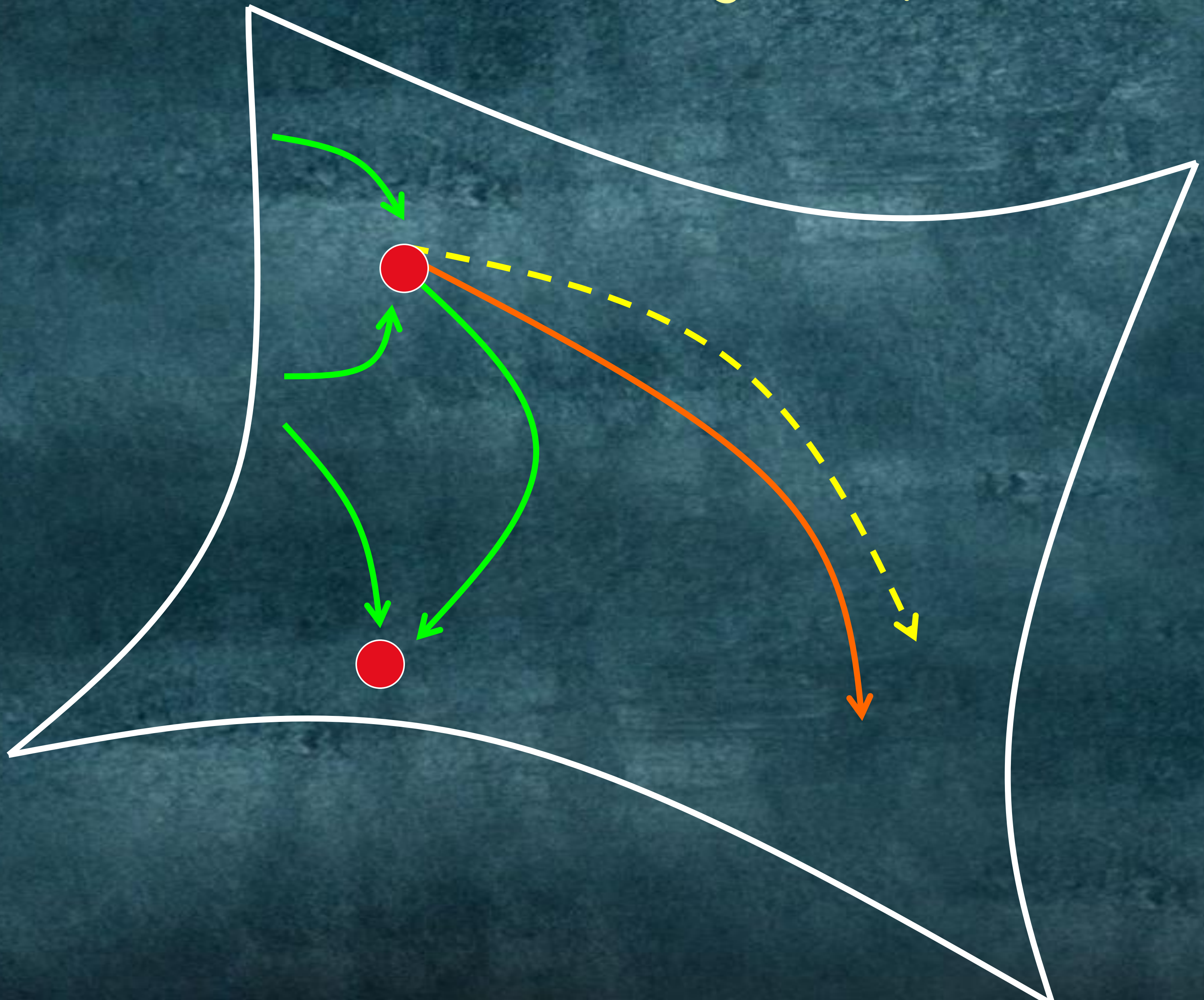


Degenerate GS
First bound state B_1
Second bound state B_2
Higher levels
$(E - E_0)/m_{\mathrm{sc},1}$
$m_{\mathrm{sc},1}R$
0
1
2
3
4
5
0
2
4
6
8
10
12
14

Close to Integrability



Form Factor Perturbation Theory

(Delfino, Simonetti, G.M.;
Delfino, G.M.;
Controzzi, G.M., ...)

$$\mathcal{A} = \mathcal{A}_{int} + g \int d^2x \, \Phi(x)$$

Form Factor Perturbation Theory

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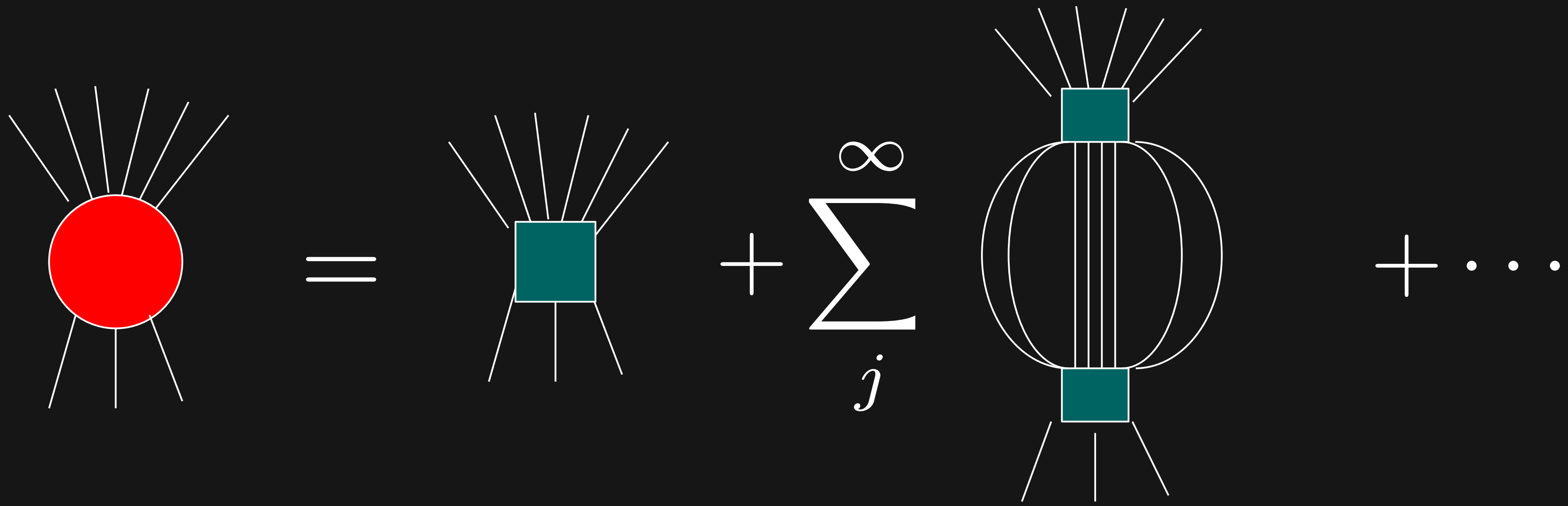


Particularly interesting is the case when we can interchange the pattern of integrability breaking

$\mathcal{A}_{int}$

$$L = \frac{1}{2}\left(\partial_\mu \Phi\right)^2 + \gamma \cos\beta\Phi + \eta\cos\omega\Phi$$

$\mathcal{A}_{int}$

Models analyzed by Form Factor Perturbation Theory

Multi-frequency Sine-Gordon (Delfino, GM)

O(3) sigma model with theta term (Controzzi, GM)

Tricritical Ising model with several couplings (Fioravanti,Simon,GM)

SUSY with metastable vacuum state (GM)

Tricks to obtain an infinite number of identities between FF

Main advantages of Form Factor Perturbation Theory

- It gives reason of the confinement of the particles, relating it to the semi-local property of the perturbing operator

- When the operator is local, the first order correctly captures the perturbed values (as shown, for instance, by comparison with numerical simulation)

  Tricks: universal ratios

Main disadvantages

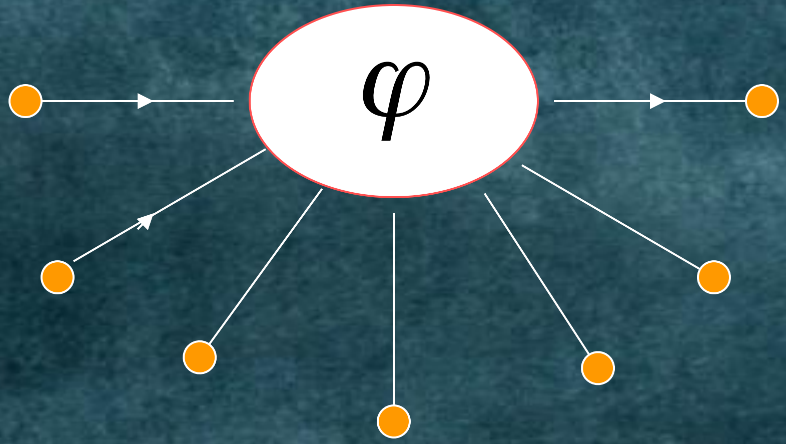
- Quite difficult to go to higher orders

Processes analyzed hereafter:

- Mass corrections and kink confinement
- Decay processes of unstable (higher mass) particles

Mass correction

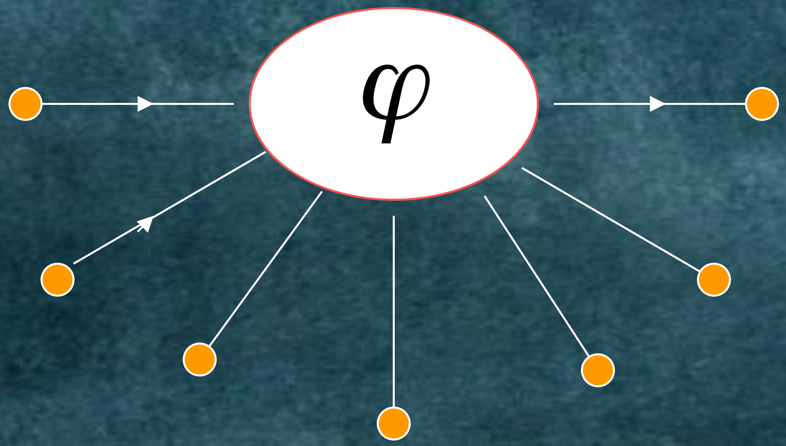
$$< 0 \,|\, \varphi \,|\, A(\beta_1) A(\beta_2) >$$



$$F_2^{\varphi}(\beta) = \frac{R}{(\beta - i\pi)} + reg$$

Mass correction

$$\delta m^2 \cong g \; < A(\beta) \,|\, \varphi \,|\, A(\beta) > = g F_2^{\varphi}(i\pi)$$



$$F_2^{\varphi}(\beta) = \frac{R}{(\beta - i\pi)} + reg$$

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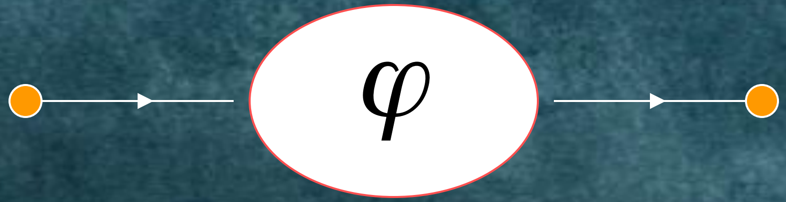
$$\delta m^2 \cong g \ < A(\beta) \,|\, \varphi \,|\, A(\beta) > = g F_2^{\varphi}(i\pi)$$

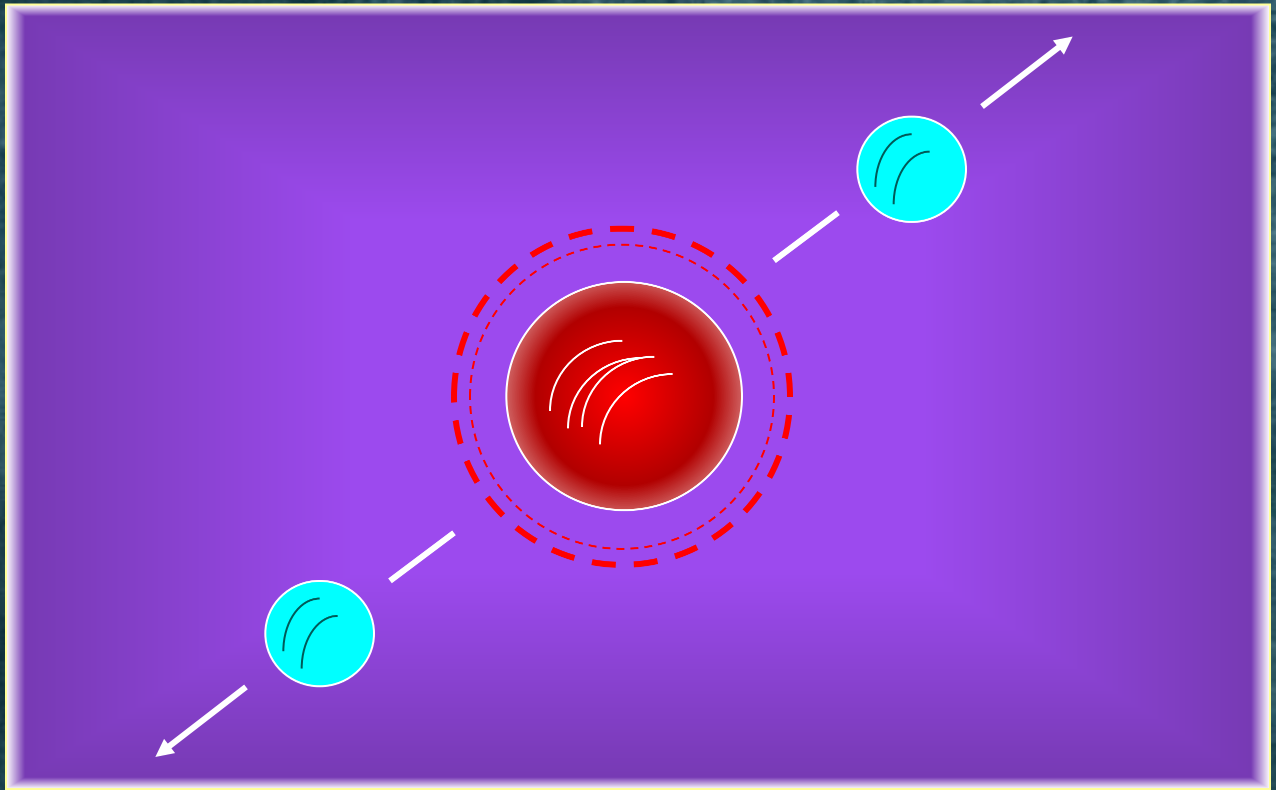


$$R \equiv -i \ \text{res}_{\beta = i\pi} F_2^{\phi}(\beta) = \left(1 - e^{2\pi i \gamma}\right) < 0 \,|\, \phi \,|\, 0 >$$

- If φ(x) is a local field (γ=0), R = 0: ADIABATIC SHIFT
- If φ(x) is a non-local field (γ≠0), R ≠ 0: CONFINEMENT

Decay process of higher mass particles

(Delfino, Grinza, GM)



(Fermi Golden Rule)

$$\Gamma \approx \frac{g^2}{2M\,|p|} \left| \langle M \,|\, \varphi(0) \,|\, m_1(p) m_2(-p) \rangle \right|^2$$

$$|p| = \sqrt{[M^2 - (m_1 - m_2)^2]\,[M^2 - (m_1 + m_2)^2]}$$

Ising Field Theory

Scaling limit of 2-d Ising Model in a magnetic field and away from the critical temperature

$$A = A_{CFT} + (T - T_c) \int d^2x \, \epsilon(x) + h \int d^2x \, \sigma(x)$$

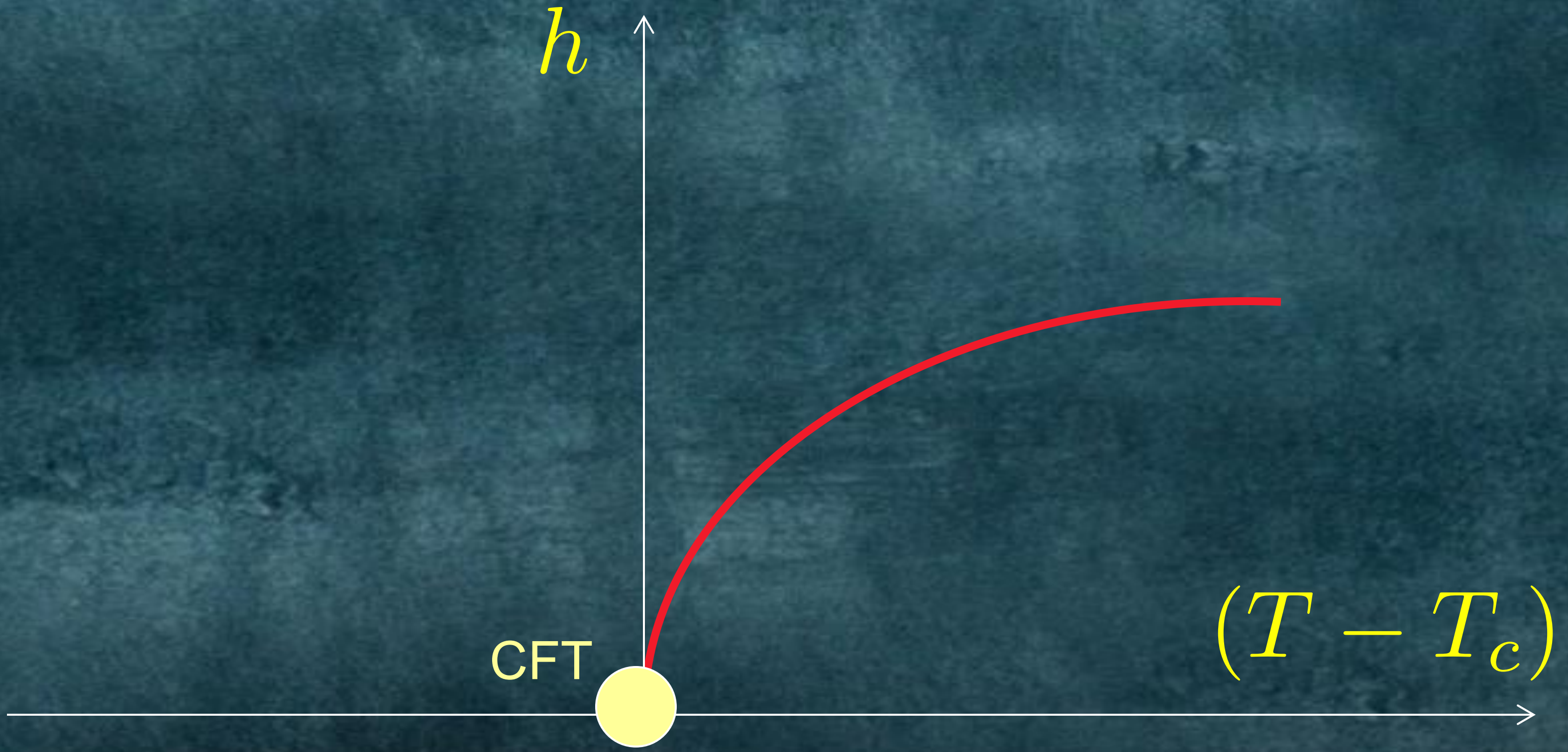
(Delfino, Simonetti, GM)

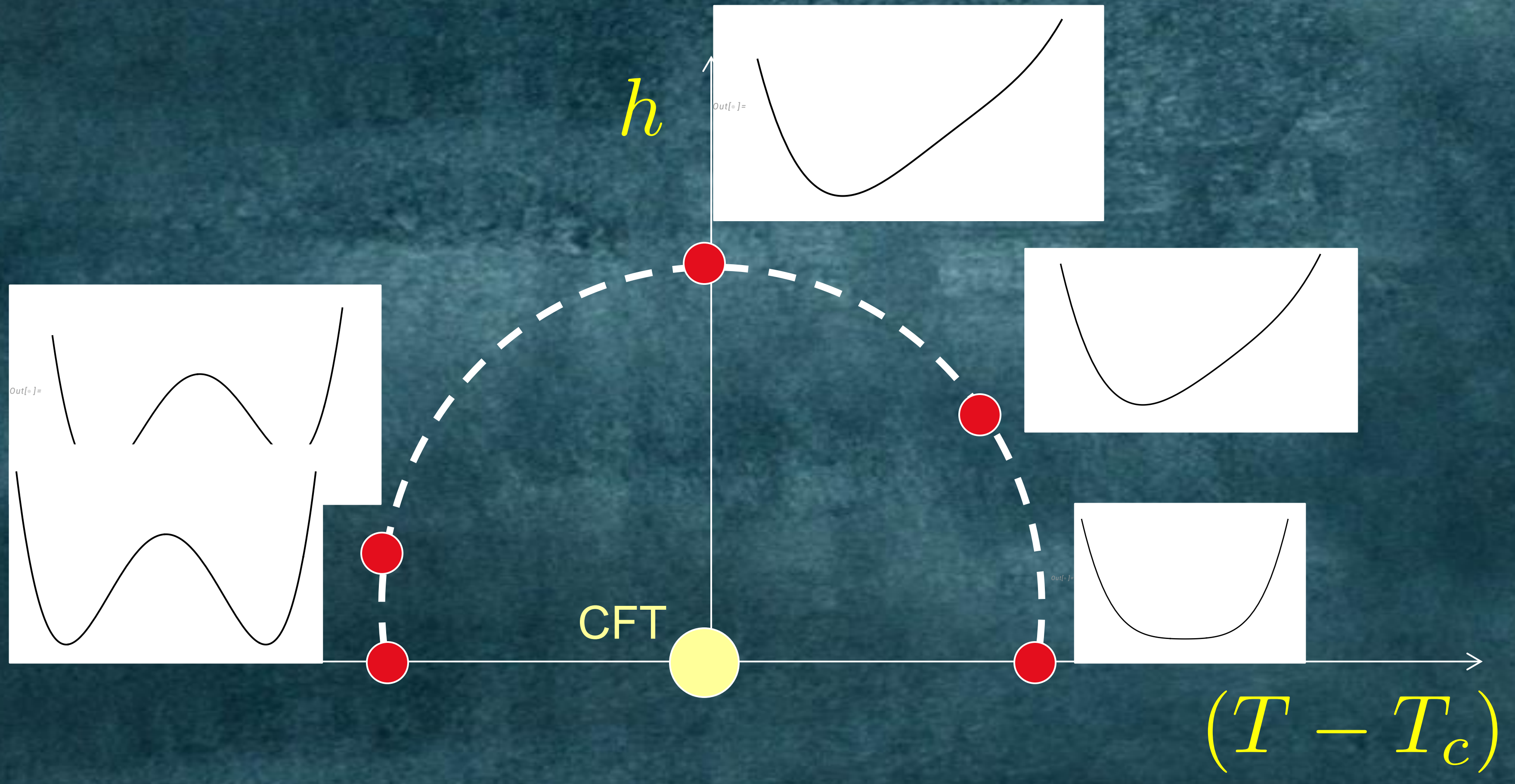
(Fonseca, Zamolodchikov)

(Grinza, Delfino, GM)

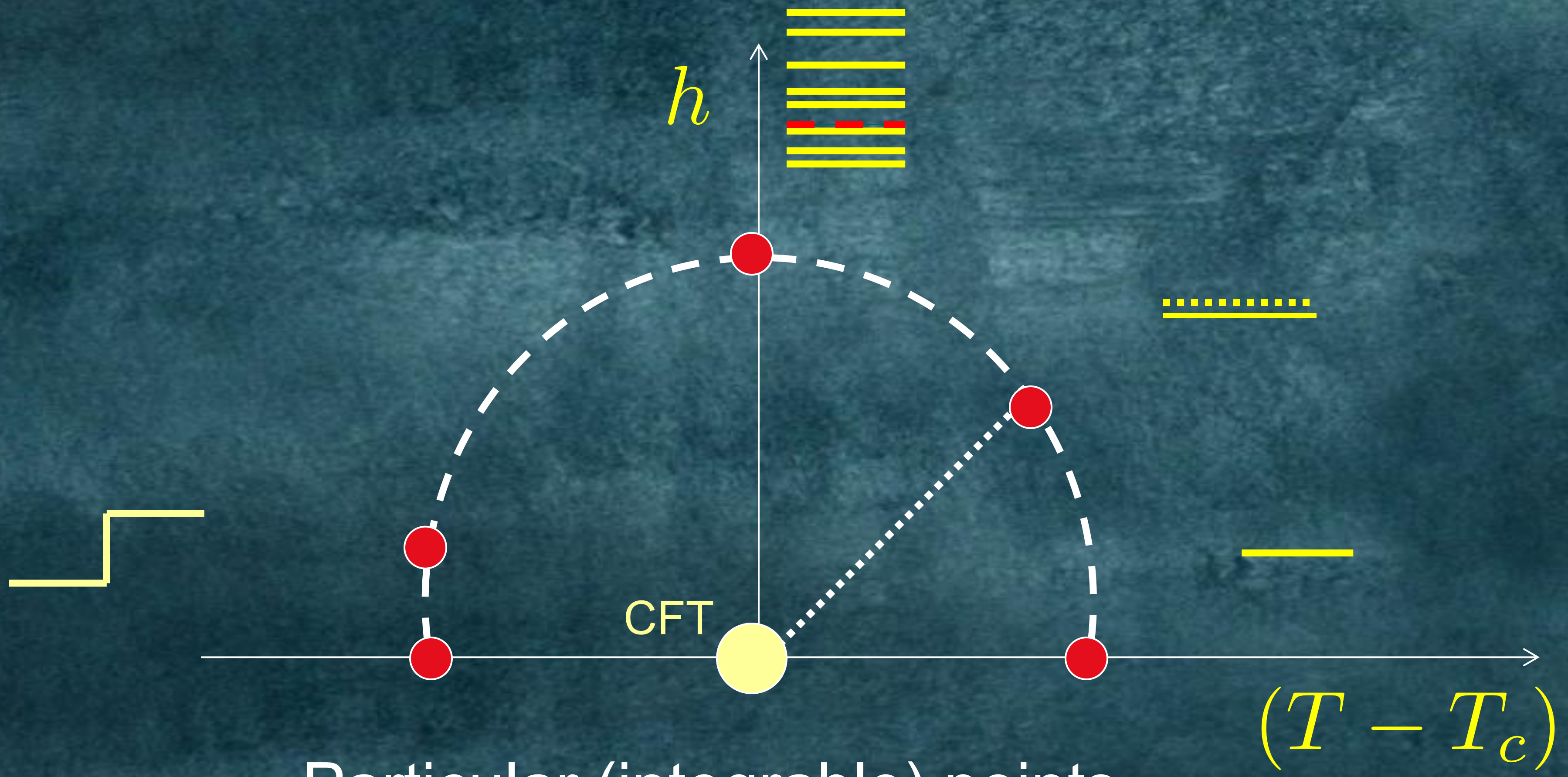
Parameter of the Model

$\chi = \tau \, |h|^{-8/15}$ labels the RG trajectories





h
CFT
$(T - T_c)$



Particular (integrable) points

h=0 → Massive free Majorana fermion

T=Tc → Massive theory with 8 particles

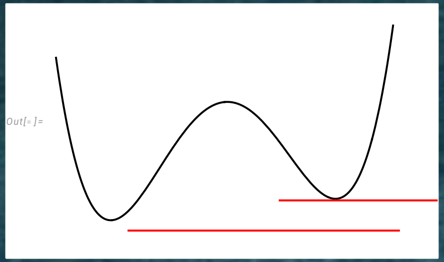
Mass Spectrum

- h=0 Free neutral fermion

$T > T_c$ ordinary particle

$T < T_c$

kink anti-kink

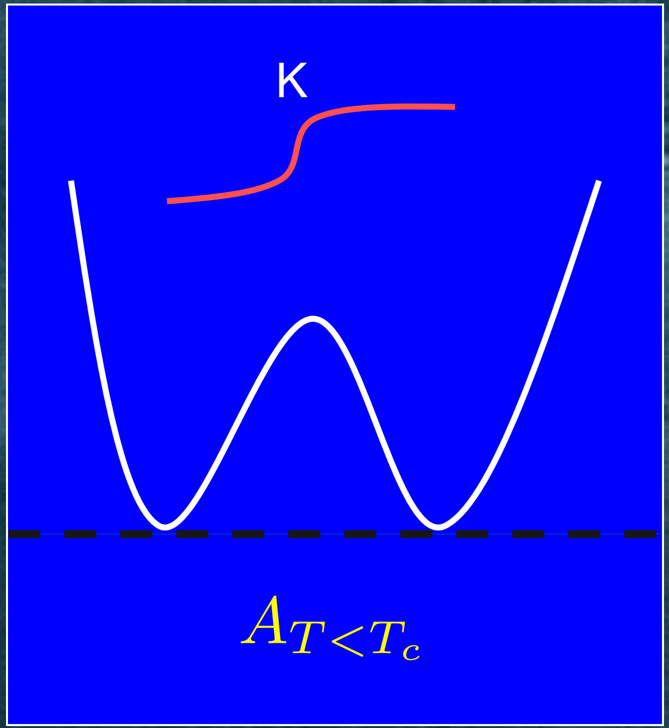


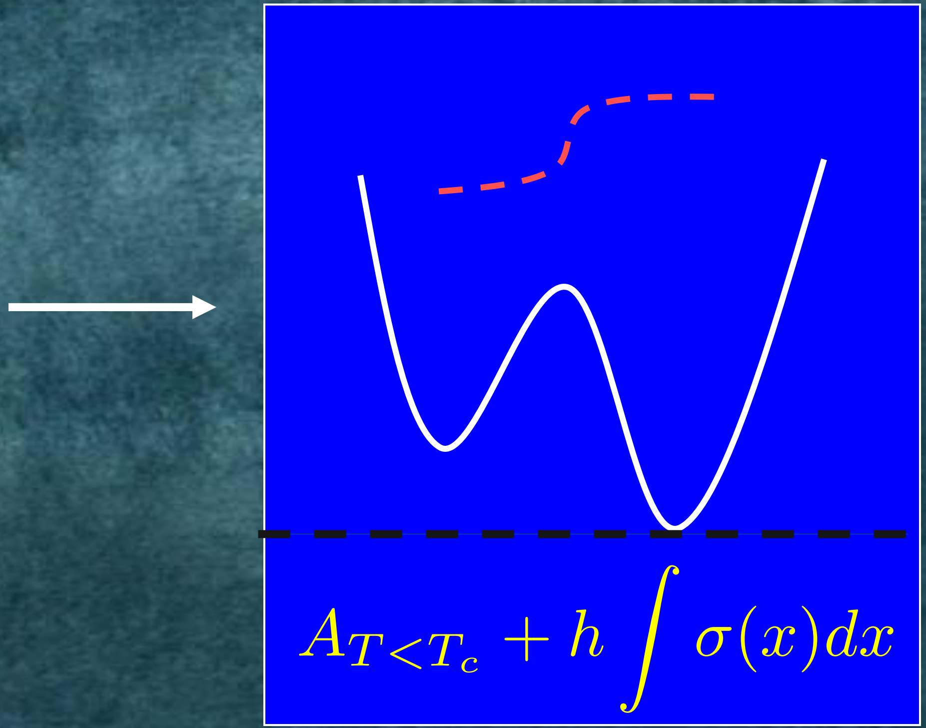
$$H = \int dx \left[\frac{1}{2}(\nabla\phi)^2 + V(\phi) \right]$$

$H[$  $] = \infty$

Kinks are confined.

In the familiar Ising model…



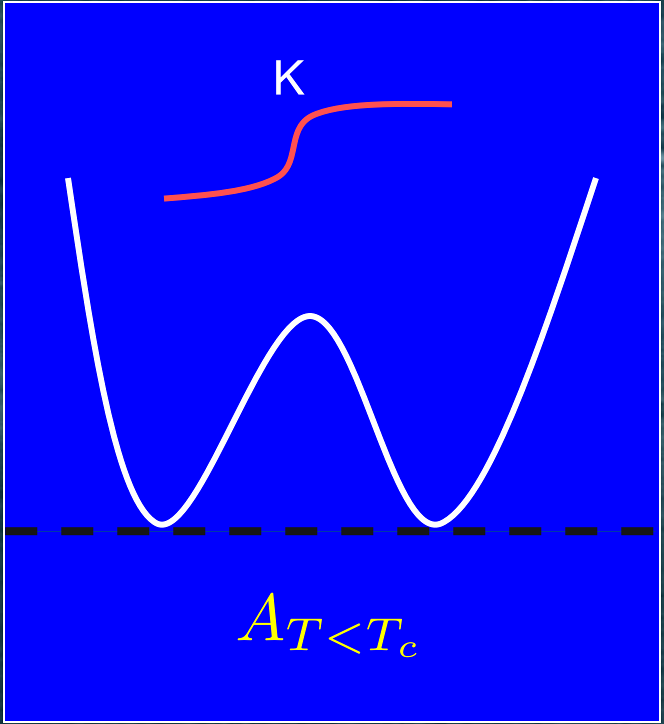


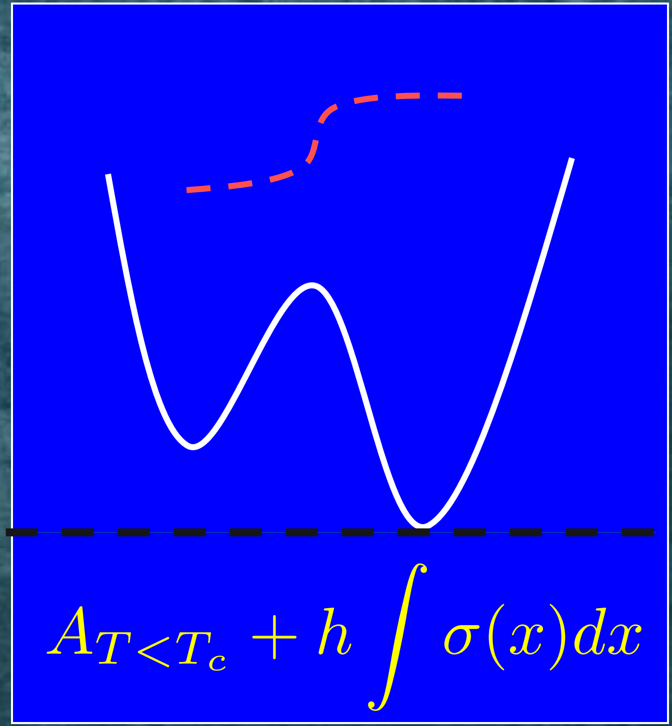
Kink excitations

Absent in the perturbed theory

$$\langle 0|\sigma(0)|K(\theta_1)K(\theta_2)\rangle = \tanh \frac{\theta_1 - \theta_2}{2}$$

In the familiar Ising model...

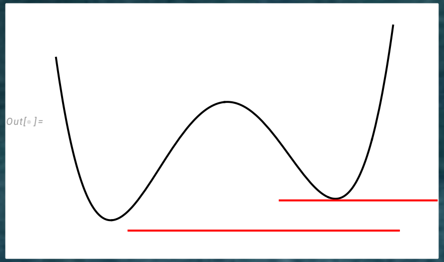




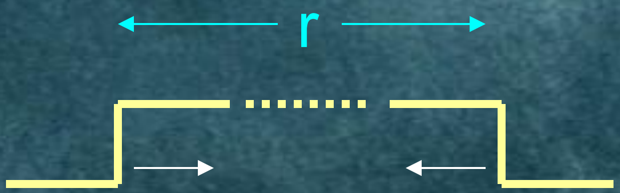
Kink excitations

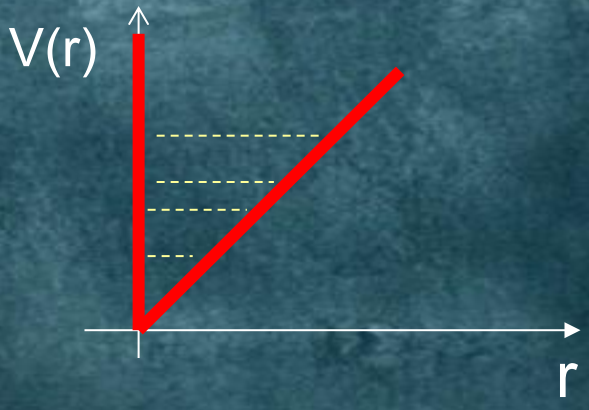
Absent in the perturbed theory

$$\delta m^2 \sim F(i\pi) = \infty$$



$$H = \int dx \left[\frac{1}{2} (\nabla \phi)^2 + V(\phi) \right]$$



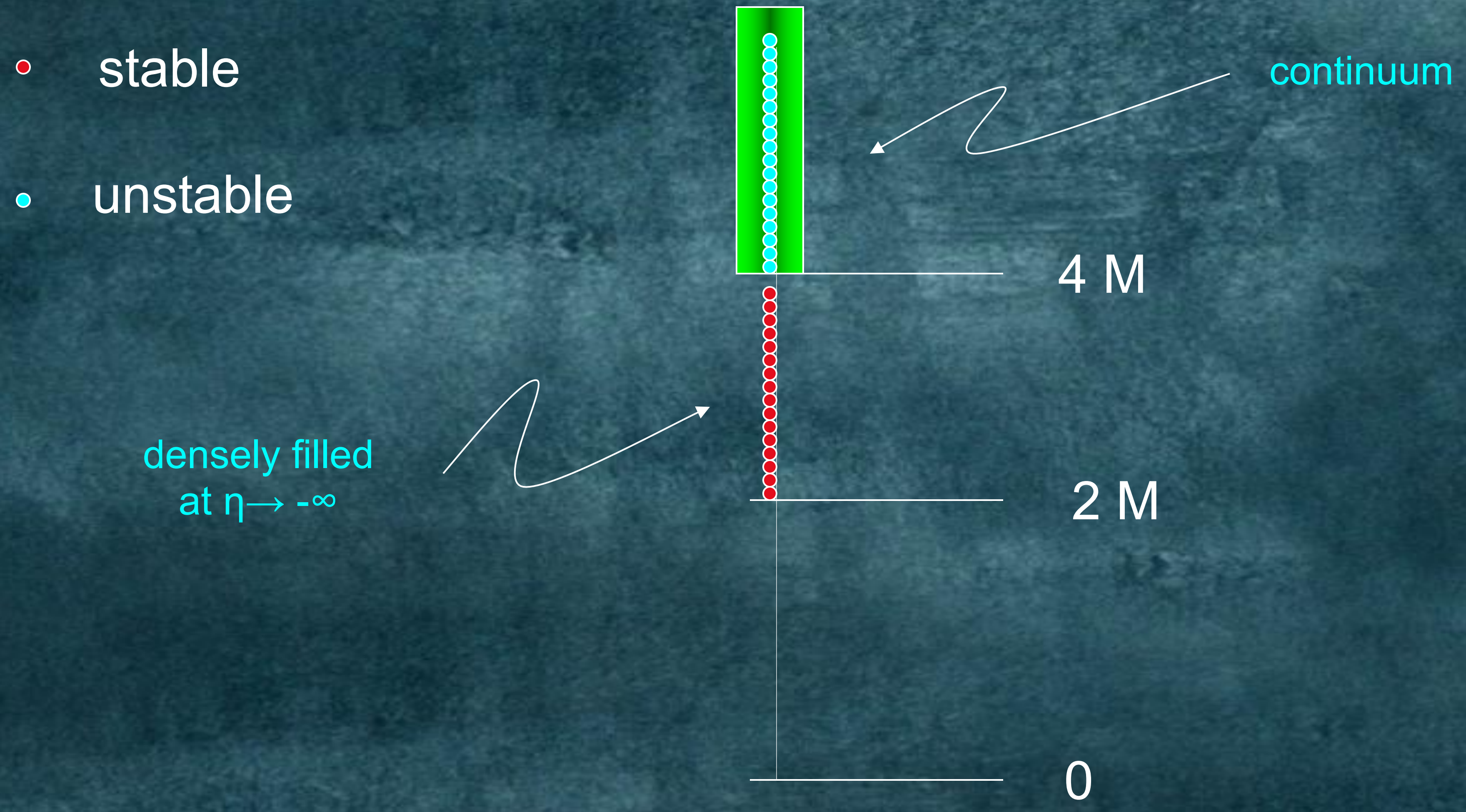


Confining potential

Kinks are confined. Spectrum given by kink-antikink bound states of mass given by

$$(2 + h^{2/3} \gamma_k^{2/3}) m$$

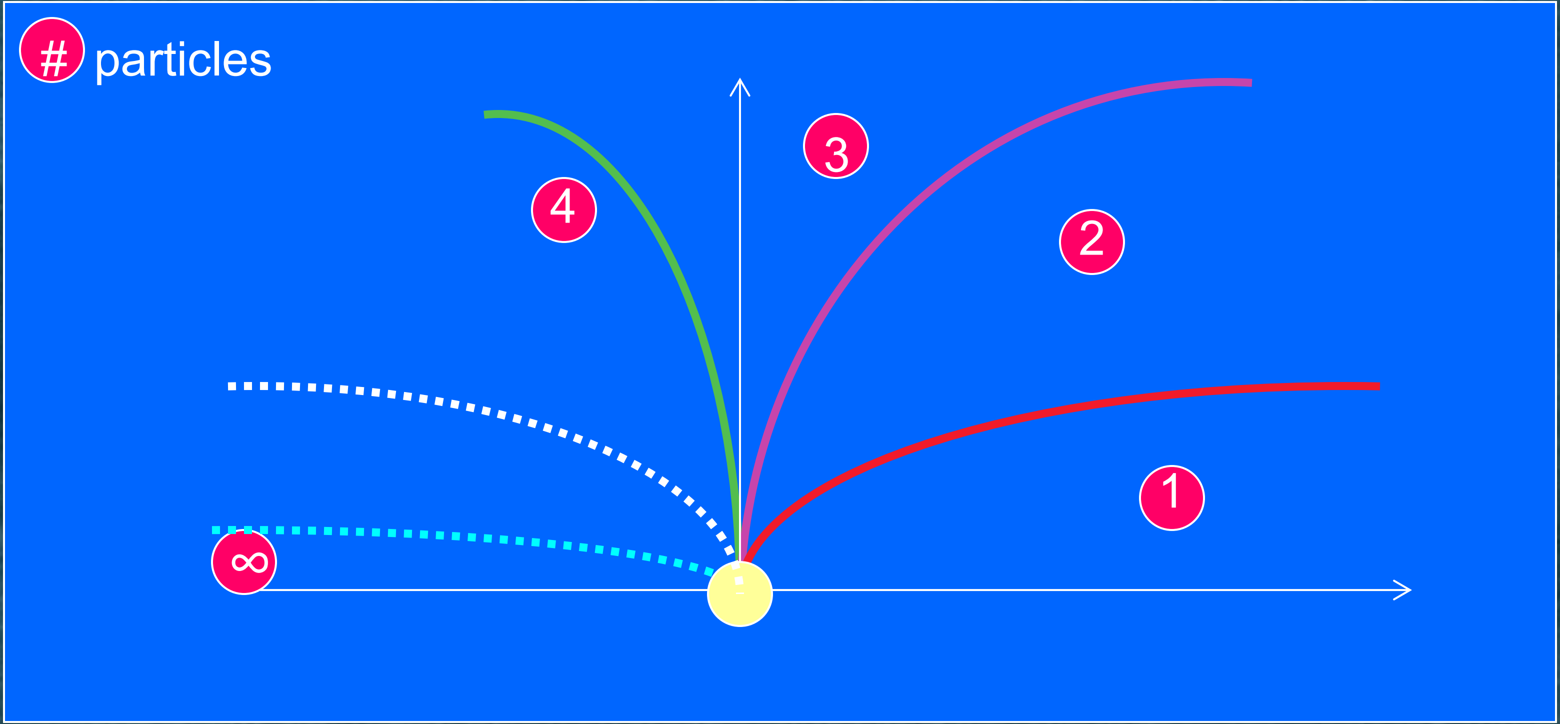
$$J_{1/3}\left(\frac{1}{3}\gamma_k\right) + J_{-1/3}\left(\frac{1}{3}\gamma_k\right) = 0$$



The number of stable particles decreases as η increases, until only one is stable at η→+∞

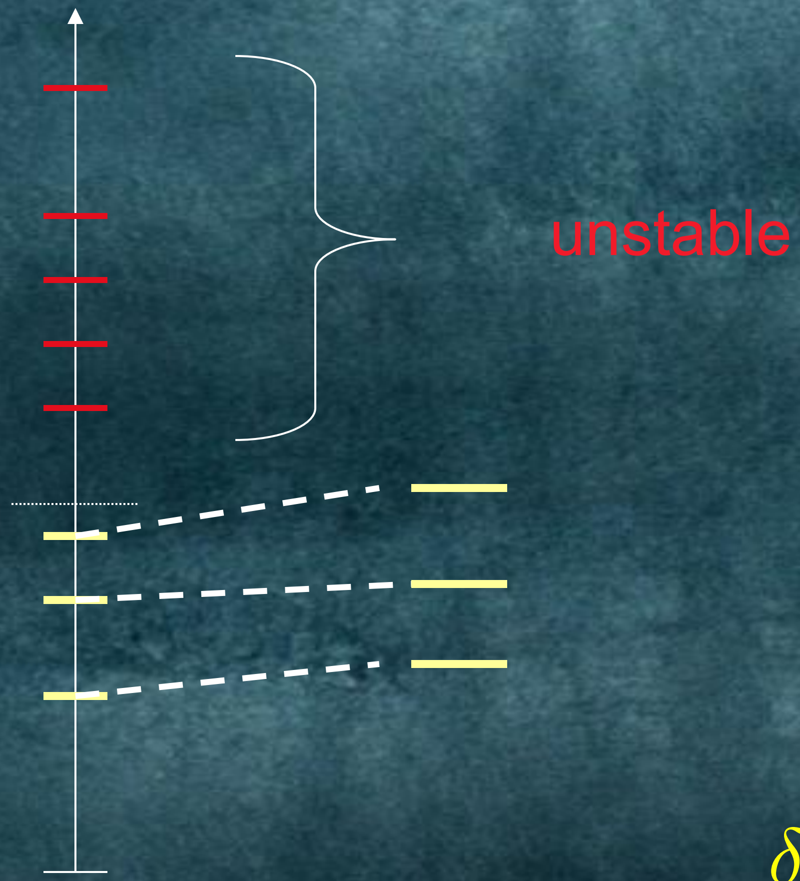
Namely, the plane is partitioned in sectors, for which there are a certain number of stable particles

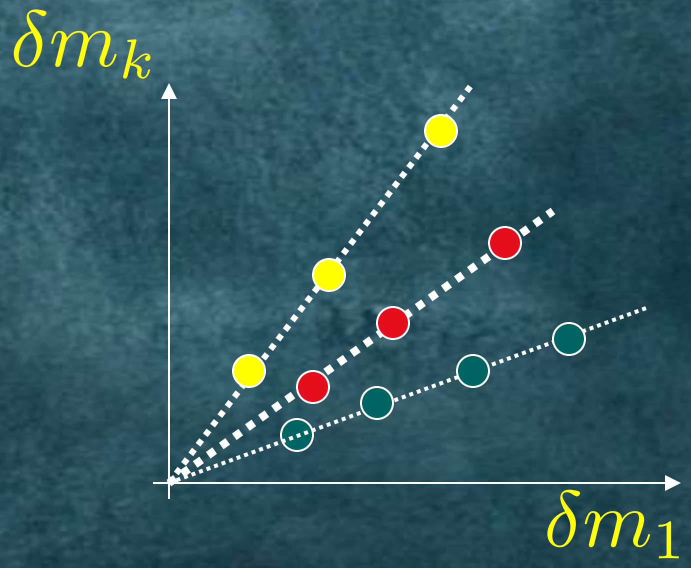
The particle m_k becomes then unstable for $\chi > \chi_k$



Mass spectrum in the vicinity of the Magnetic Axis with temperature different from the critical value

(Delfino, Simonetti, GM)





$$\frac{\delta \epsilon_0}{\delta m_1} = -0.0558 m_1$$

$$\frac{\delta m_2}{\delta m_1} = 0.8616 \qquad \frac{\delta m_3}{\delta m_1} = 1.508$$

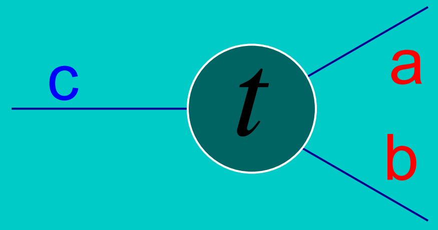
Decay channels of higher mass particles

$$A_4 \to A_1\, A_1$$

$$A_5 \to A_1 A_1 \;,\; A_1\, A_2$$

$$A_6 \to A_1\, A_1 \;,\; A_1\, A_2 \;,\; A_1\, A_3$$

f_{abc}



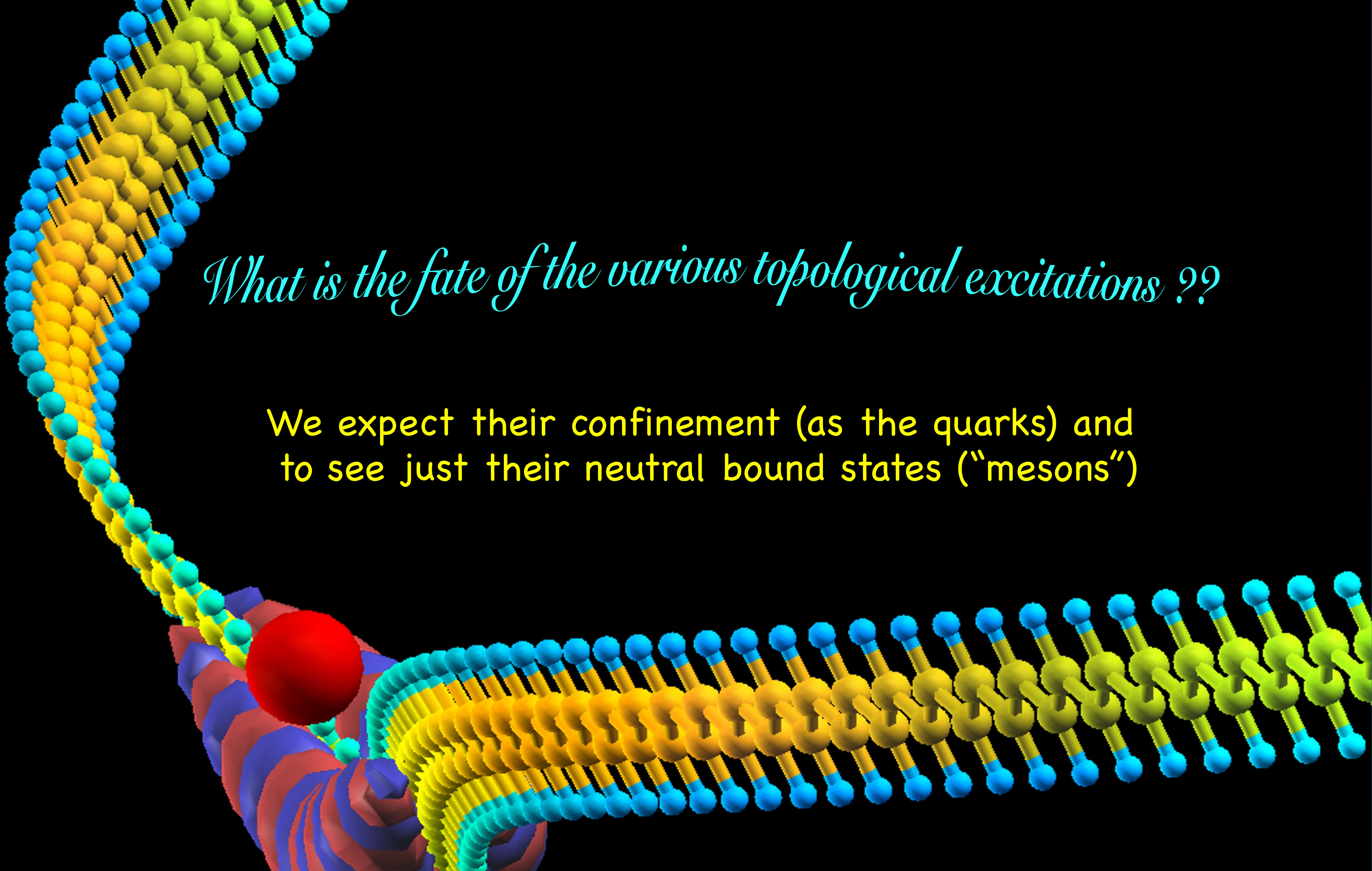
Decay widths of higher particles

$$\frac{\Gamma_4}{\Gamma_5} = 4.287...$$

- Alias, the particle with higher mass lives 4 time longer than the one with lower mass!

- Branching ratios of the decay of A_5

$A_5 \to A_1\, A_1$	47%
$A_5 \to A_1\, A_2$	53%

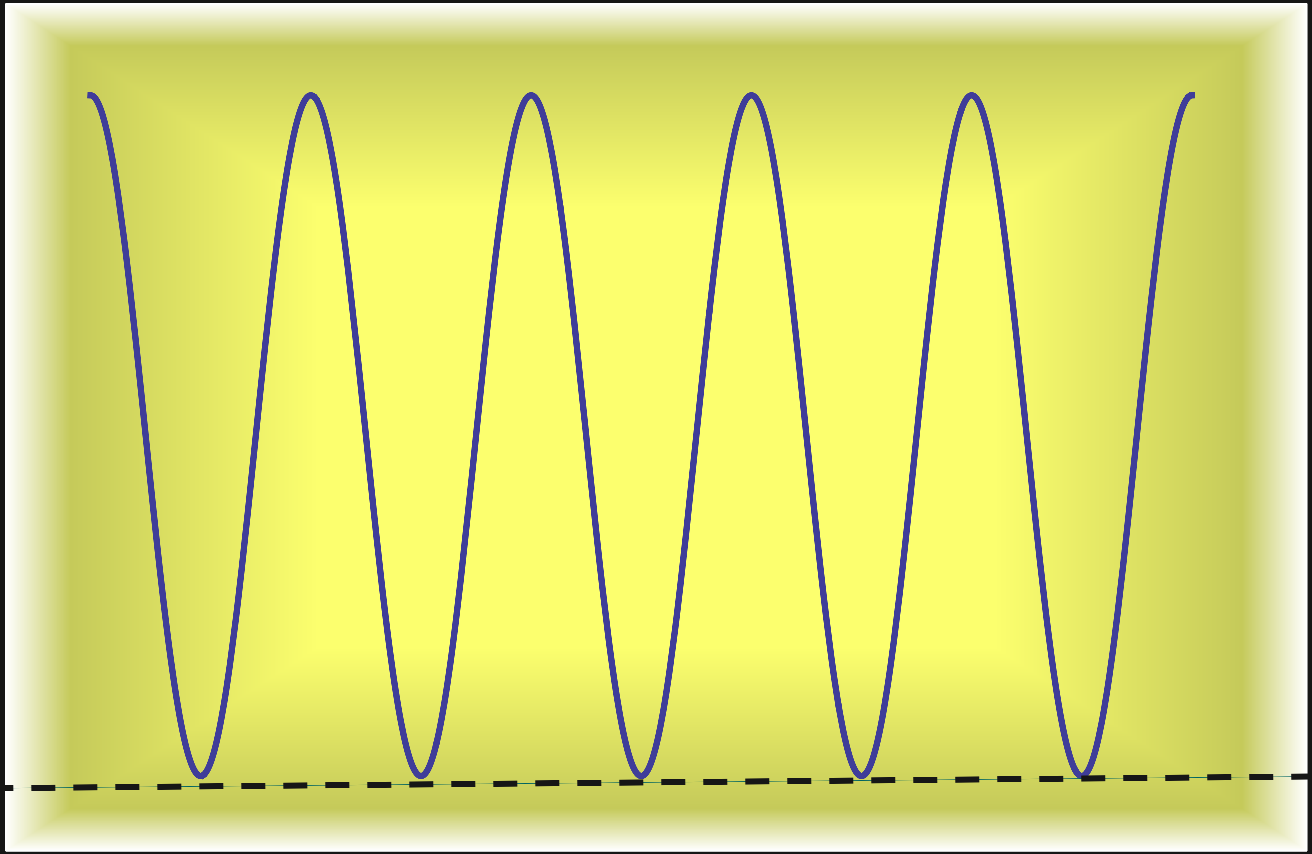


What is the fate of the various topological excitations ??

We expect their confinement (as the quarks) and to see just their neutral bound states ("mesons")

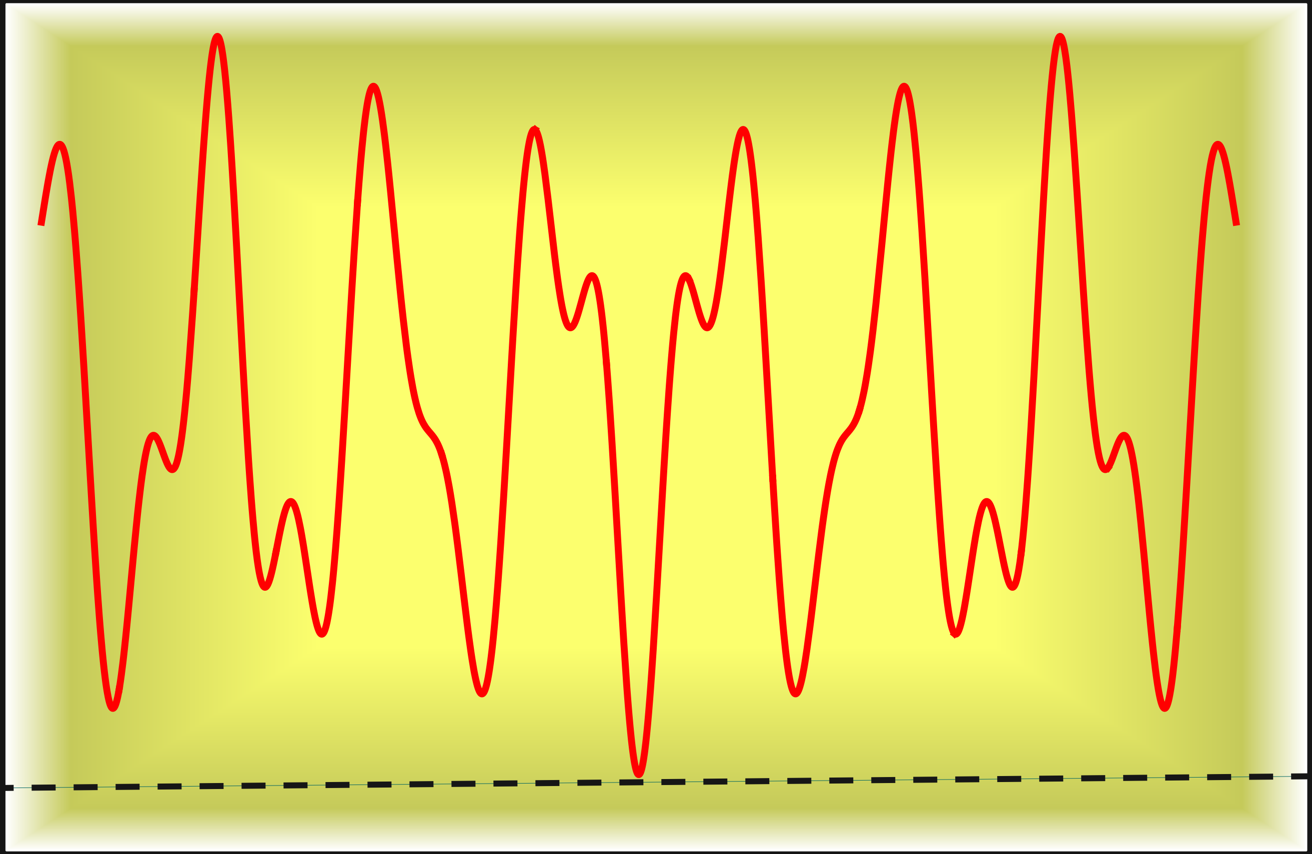
Fragility of the kink in purely bosonic theories

$$L = \frac{1}{2}\left(\partial_\mu \Phi\right)^2 + \gamma \cos\beta\,\Phi$$



Fragility of the kink in purely bosonic theories

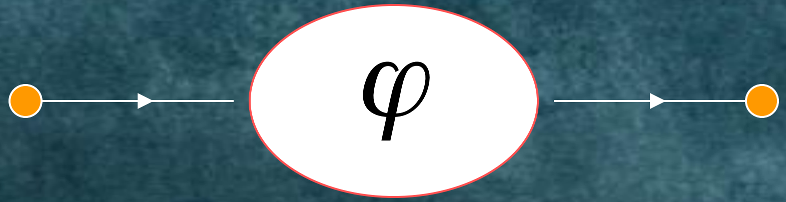
$$L = \frac{1}{2}\left(\partial_\mu \Phi\right)^2 + \gamma \cos \beta \Phi + \eta \cos \omega \Phi$$



gone !

Mass correction

$$\delta m^2 \cong g \; < A(\beta) \,|\, \varphi \,|\, A(\beta) > = g F_2^{\varphi}(i\pi)$$



$$R \equiv -i \;\, res_{\beta=i\pi} F_2^{\phi}(\beta) = \left(1 - e^{2\pi i \gamma}\right) <0|\phi|0>$$

- If φ(x) is a local field (γ=0), R = 0: ADIABATIC SHIFT
- If φ(x) is a non-local field (γ≠0), R ≠ 0: CONFINEMENT

Semi-local index of vertex operators in the SG model

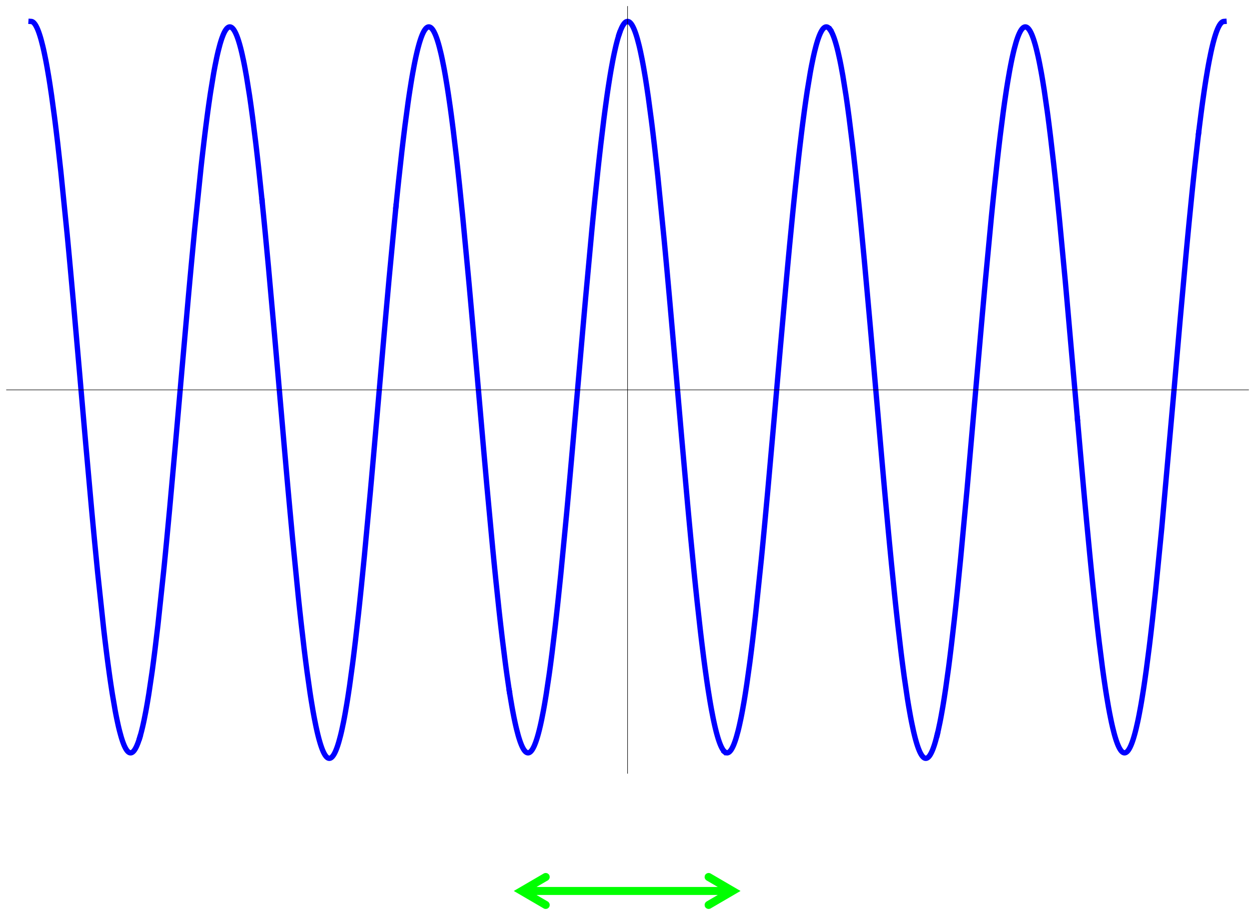
$$L = \frac{1}{2}(\partial\varphi)^2 - \lambda \cos \beta\varphi$$

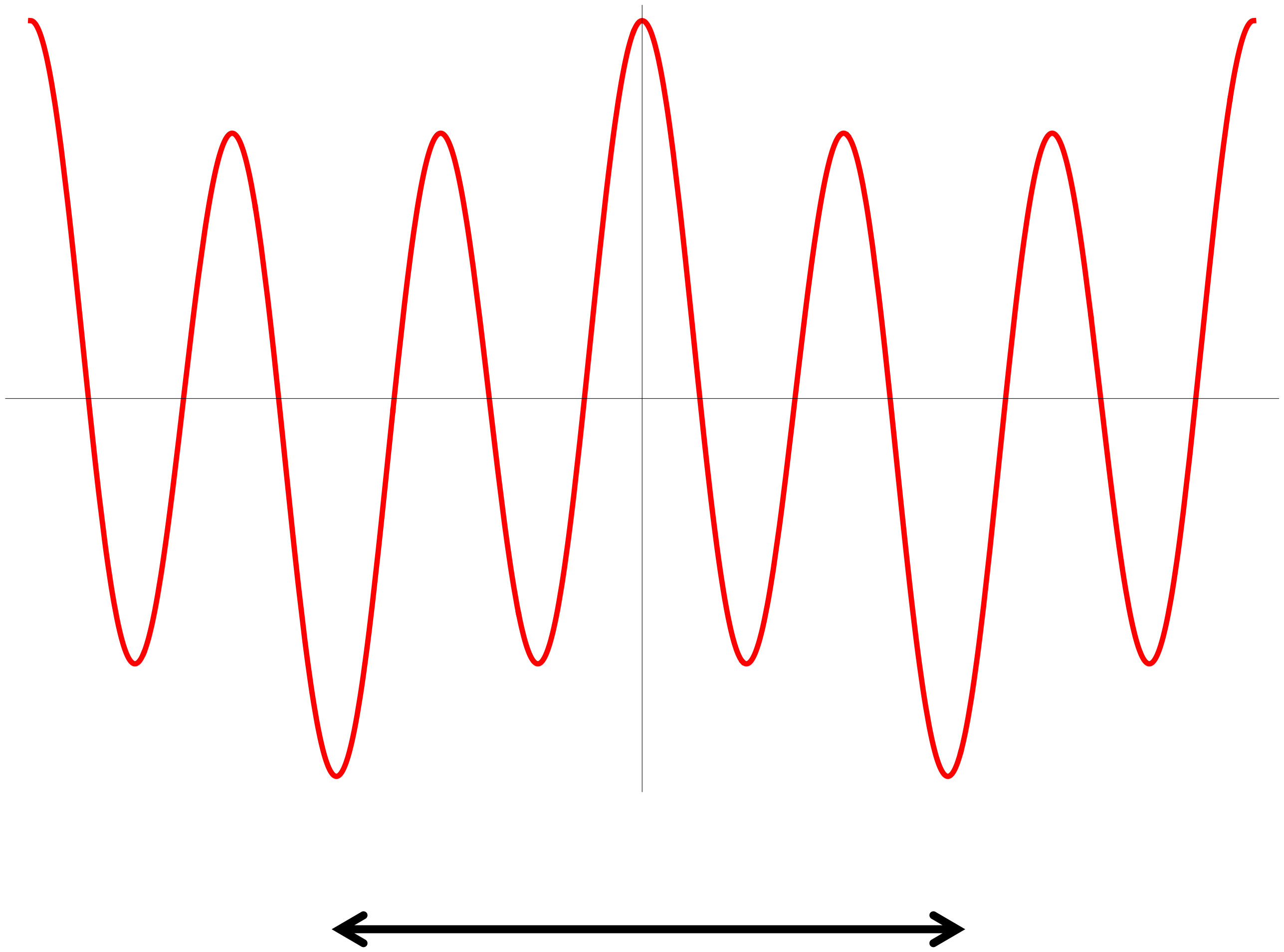
$$O = \cos \alpha\varphi \qquad \gamma = \frac{\alpha}{\beta}$$

$$\mathrm{Res}_{\beta=i\pi} F_2^{\mathcal{O}} = [1 - \cos(2\pi\gamma)] \langle 0|\mathcal{O}(0)|0\rangle$$

- If γ is irrational, no solitons survive
- If $\gamma = \frac{m}{n}$ the original soliton confines but "packages" made of

  n original solitons survive as stable excitations





While kink excitations are rather fragile objects in purely bosonic theory,

they can be instead stable excitations in integrability breaking that takes place in SUSY theories

"Fermions stabilize the kinks"

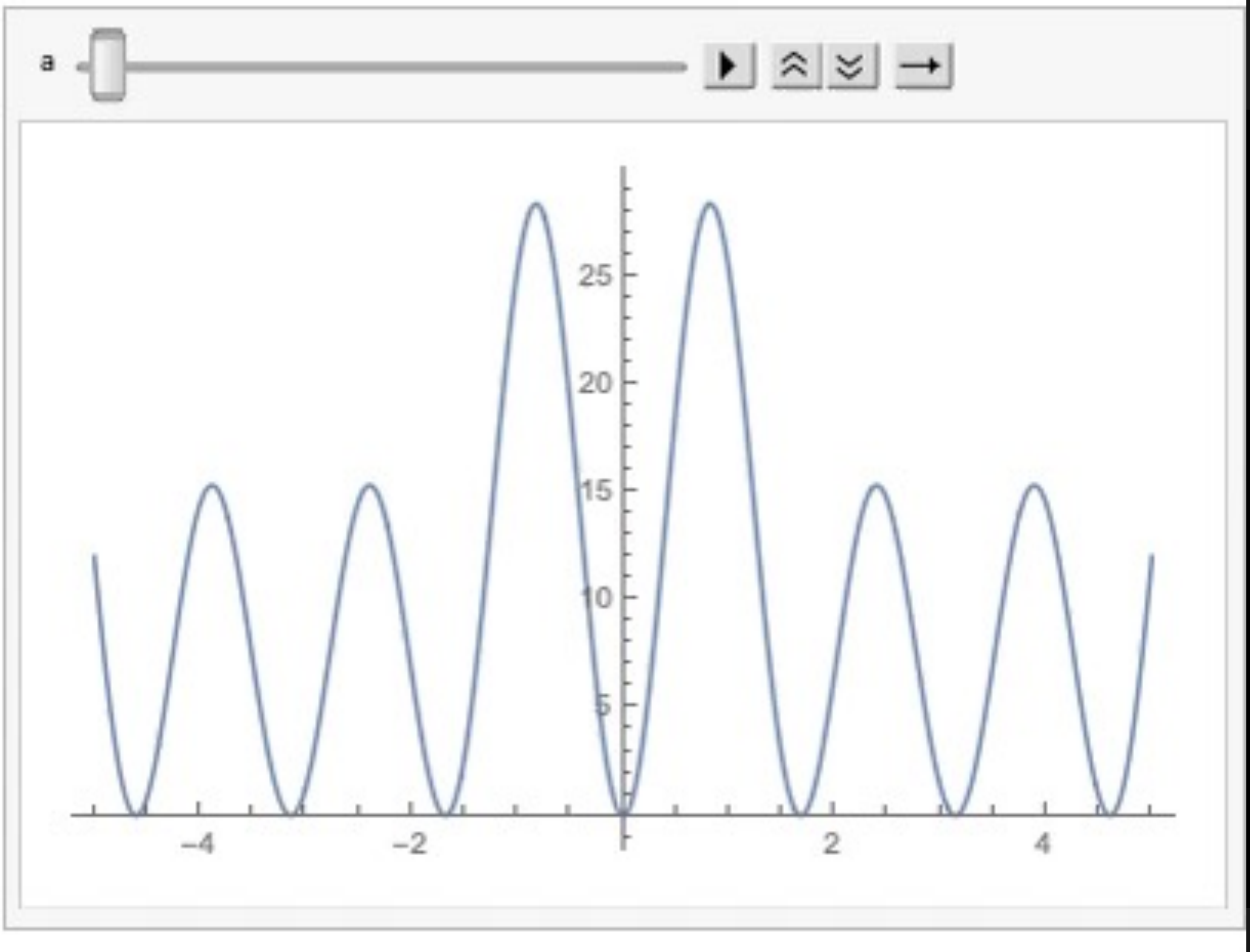
Multi-frequency Super Sine-Gordon

(GM, JHEP)

$$A = \int dx d\theta \left[\frac{1}{4} (\bar{D}\Phi)(D\Phi) - \underbrace{\cos\Phi - \lambda\cos(\omega\Phi)}_{W(\Phi)} \right] \qquad \omega = \frac{p}{q}$$

$$[W'(\varphi)]^2 = [\sin\varphi + \lambda\sin(\omega\varphi)]^2$$

- For all values of λ the origin is always a zero, i.e. SUSY exact
- At lowest order, the vacua of the original potential continue to remain degenerate.


a
25
20
15
10
5
−4
−2
2
4

Multi-frequency Super Sine-Gordon

(GM, JHEP)

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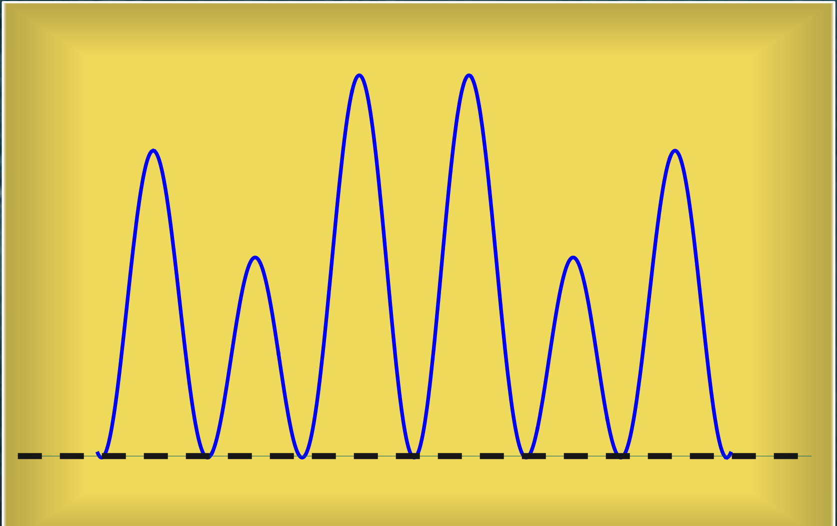
- At lowest order, the vacua of the original potential continue to remain degenerate. The kinks are **stable** at weak coupling!



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Form Factor Perturbation Theory

$$\Upsilon = \sin\varphi \sin\omega\varphi + \omega \bar{\psi} \psi \cos\omega\varphi$$

$$\mathrm{Re}\, s_{\theta = \pm i\pi} F^{\Upsilon}_{S\bar{S}}(\theta) = \left[1 + \cos(\pi\omega)\right] < 0 | \Upsilon | 0 >$$

= 0 !

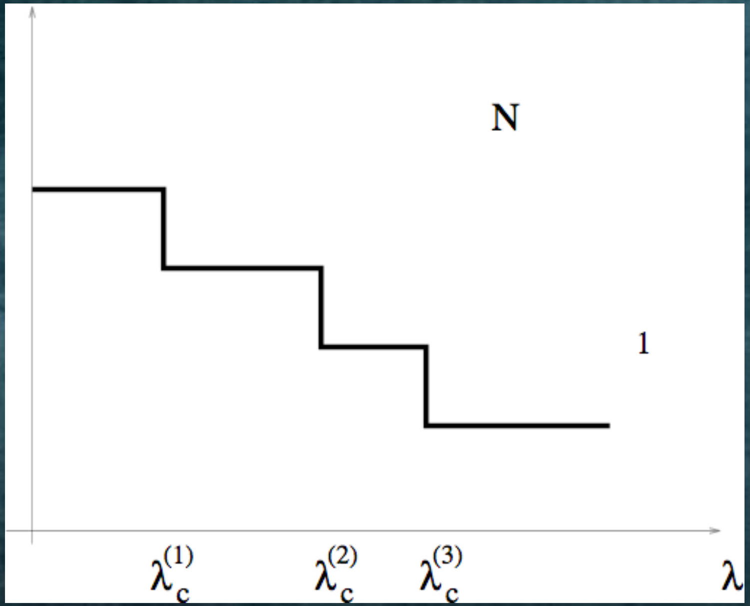
What happens at strong coupling?

$$(W'(\varphi))^2 = \left[\sin\varphi + \lambda \sin\left(\frac{p}{q}\varphi\right)\right]^2$$

N(λ)= Number of zeros in (0, 2 π q)

$$N(0) = 2q + 1$$
$$N(\infty) = 2p + 1$$

$$\Delta N = 2(q - p)$$



What happens at strong coupling?

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$N(\lambda)$= Number of zeros in $(0, 2\pi q)$

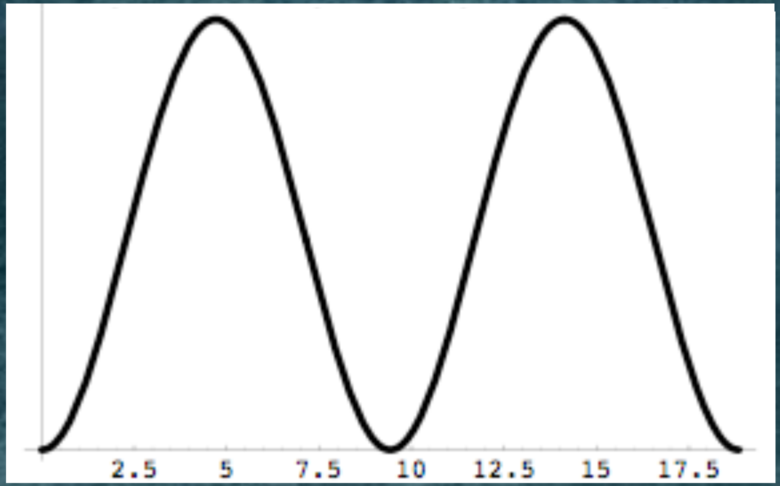
$$N(0) = 2q + 1$$
$$N(\infty) = 2p + 1$$

$$\Delta N = 2(q - p)$$

- Since the kinks own their existence to the zeros, a variation of their number implies that some of them should **disappear**
- For their topological nature, the disappearance of a kink signals a **phase transition**

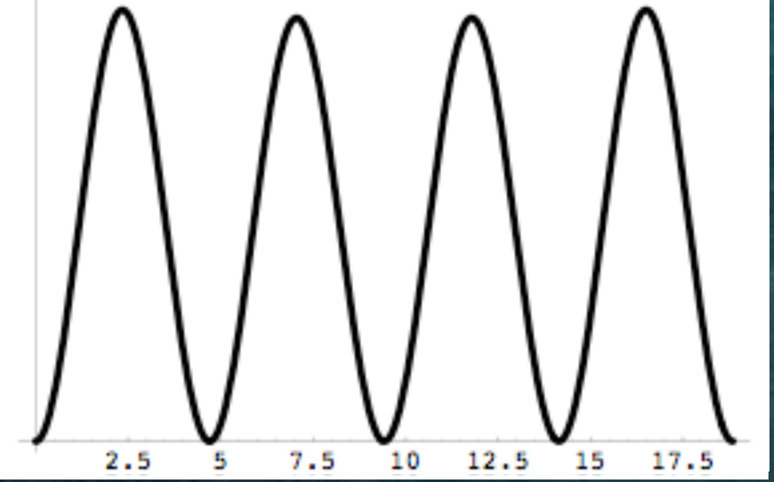
When **(q-p) = even**, there will be a sequence of phase transitions that will recall the one of Tricritical Ising -> Ising.

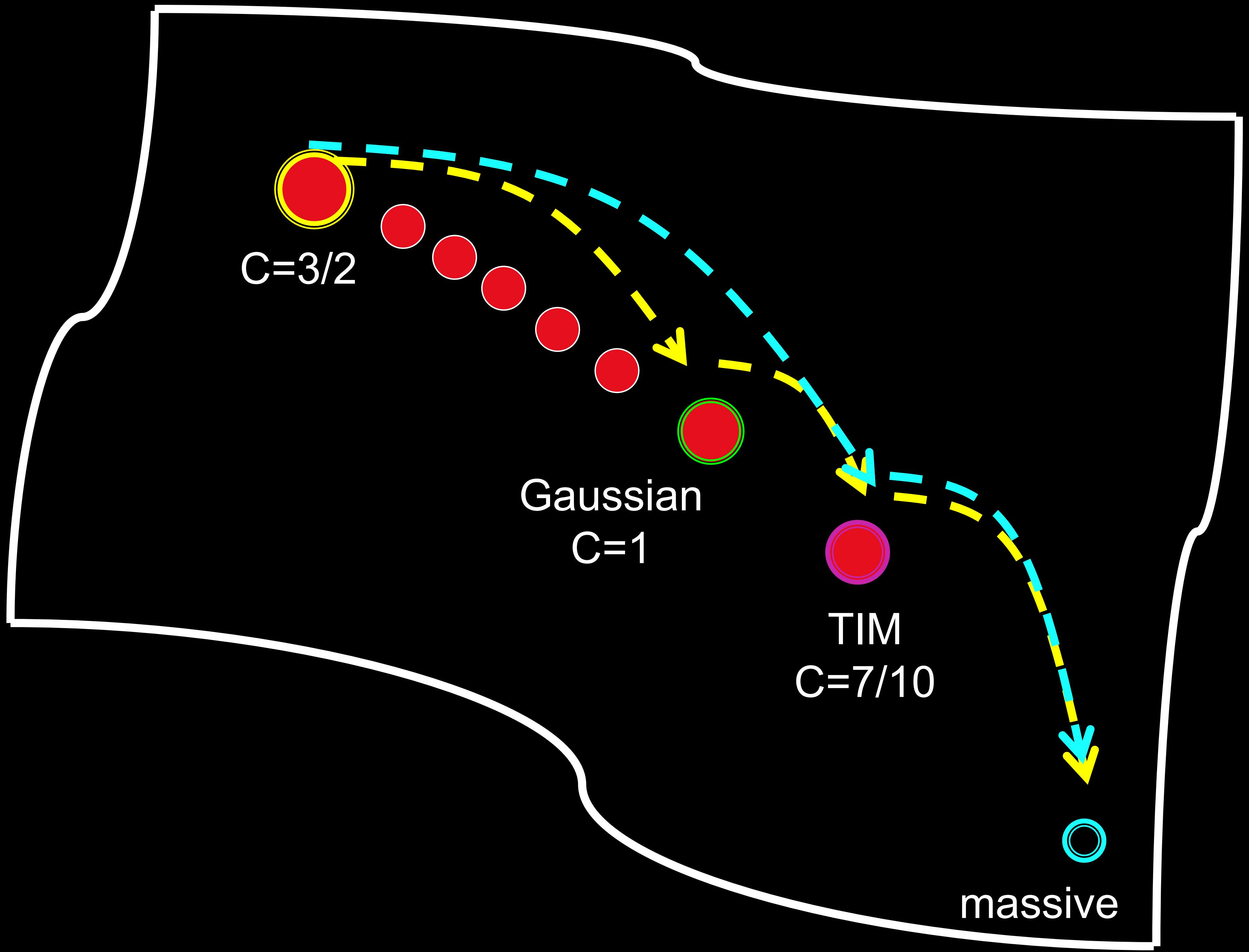
Local SUSY breaking, with meta-stable vacua



When **(q-p) = odd**, there will be, in addition to a sequence of phase transitions TIM ->Ising, also a phase transition that will will recall the one of the **gaussian model**.

In this case, there is a vacuum where **SUSY is exact** before and after the phase transition




C=3/2
Gaussian
C=1
TIM
C=7/10
massive

Conclusions and Perspectives

- There is a rather robust knowledge on how to control and predict the spectrum in a generic 2-dim QFT

- In the perspective to explore their experimental study, particularly interesting is the pattern of kink excitations in multi Sine-Gordon theories

- Equally interesting is the possibility to realize experimentally SUSY theories through mixture and to check in particular the persistency of the kink excitations