Application of hidden fermionic structure to the integrable QFT

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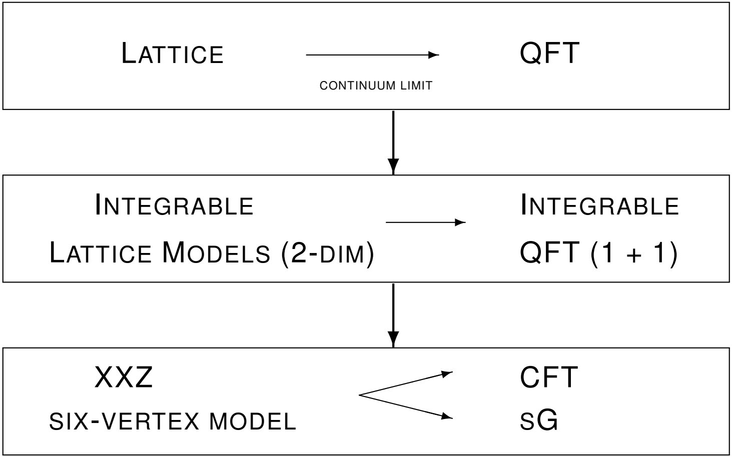
Integrable Systems and Field Theory, Paris, LPTHE
October 11, 2023

Lattice–QFT correspondence



XXZ vs. CFT

- study of the low-lying excitations of the XXZ Hamiltonian

Blöte, Cardy, Nightingale (85), Affleck (85), Alcaraz, Martins (88)

and finite-size corrections Luther, Peschel (75), Lukyanov (98) etc.

$$\mathcal{H}_{\mathrm{xxz}} = \frac{1}{2}\sum_{j=1}^{N}\Big(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \frac{1}{2}(q+q^{-1})\,\sigma_j^z\sigma_{j+1}^z\Big)$$

deformation parameter: $q = e^{\pi i\nu}, \quad \frac{1}{2} < \nu < 1$

twisted boundary conditions: $\sigma_{N+1}^{\pm} = e^{\pm i\Phi}\sigma_1^{\pm}, \quad \sigma_{N+1}^z = \sigma_1^z$

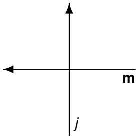
periodic b.c.: $\Phi = 0$

with the ground-state energy in the limit $N \to \infty$

$$E_0(N) = e_\infty N - \frac{\pi c}{6}N^{-1} + O(N^{-2}), \quad c = 1 - \frac{6\Phi^2}{\pi^2(1-\nu)}$$

The six-vertex model, Matsubara space and monodromy matrix

It is well known that the XXZ model is related to the six-vertex model.
Integrable structure is generated by R-matrix or L-operator



$$\equiv \quad L_{j,\mathbf{m}}(\zeta) = q^{-\frac{1}{2}\sigma_j^z\sigma_{\mathbf{m}}^z} - \zeta^2 q^{\frac{1}{2}\sigma_j^z\sigma_{\mathbf{m}}^z} - \zeta(q-q^{-1})(\sigma_j^+\sigma_{\mathbf{m}}^- + \sigma_j^-\sigma_{\mathbf{m}}^+)$$

Consider the XXZ-model in infinite volume. Space of states is $\mathfrak{H}_{\mathrm{S}} = \bigotimes_{j=-\infty}^{\infty} \mathbb{C}^2$

Also we introduce the Matsubara space: $\mathfrak{H}_{\mathbf{M}} = \bigotimes_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}} \mathbb{C}^2$

and monodromy matrix: $T_{\mathrm{S},\mathbf{M}} = \overset{\curvearrowright}{\prod_{j=-\infty}^{\infty}} T_{j,\mathbf{M}}\,,\ \ T_{j,\mathbf{M}} \equiv T_{j,\mathbf{M}}(1),\ \ T_{j,\mathbf{M}}(\zeta) = \overset{\curvearrowleft}{\prod_{\mathbf{m}=\mathbf{1}}^{\mathbf{n}}} L_{j,\mathbf{m}}(\zeta/\tau_{\mathbf{m}})$

In homogeneous case we take $\tau_{\mathbf{m}} = q^{1/2}$.

The local operators

Introduce a local operator $\mathcal{O}$ on $\mathfrak{H}_\mathrm{S}$ which acts non-trivially only on a finite segment of $\mathfrak{H}_\mathrm{S}$. We call quasi-local operator with tail α the following product

$$q^{2\alpha S(0)}\mathcal{O}$$

Here we defined

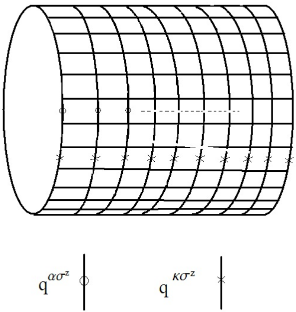
$$S(k)=\frac{1}{2}\sum_{j=-\infty}^{k}\sigma_j^z$$

So, $S(0)$ acts on the semi-infinite chain and $S=S(\infty)$ is the total spin.
We call

$$q^{2\alpha S(0)}$$

lattice 'primary field' and parameter α – **disorder field.**

Partition function on cylinder



Cut corresponds to

insertion of local operator $\mathcal{O}$

κ – **“imaginary” magnetic field**

α – **disorder field**

Matsubara expectation values:

$$Z_{\mathbf{n}}\left\{q^{2\alpha S(0)}\mathcal{O}\right\} = \frac{\mathrm{Tr}_{\mathrm{S}}\mathrm{Tr}_{\mathbf{M}}\left(T_{\mathrm{S},\mathbf{M}}\ \ q^{2\kappa S+2\alpha S(0)}\mathcal{O}\right)}{\mathrm{Tr}_{\mathrm{S}}\mathrm{Tr}_{\mathbf{M}}\left(T_{\mathrm{S},\mathbf{M}}\ \ q^{2\kappa S+2\alpha S(0)}\right)}$$

Fermionic operators

Describe the basis of quasi-local operators via certain creation operators. Jimbo, Miwa, Smirnov, Takeyama, HB (07–09)

Creation operators $\mathbf{t}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$ together with annihilation operators $\mathbf{b}, \mathbf{c}$ are constructed with help of representation theory of quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ and act in space

$$\mathcal{W}^{(\alpha)} = \bigoplus_{s=-\infty}^{\infty} \mathcal{W}_{\alpha-s,s}$$

where $\mathcal{W}_{\alpha-s,s}$ is subspace of quasi-local operators of the spin s. They are formal power series of $\zeta^2 - 1$ and have the block structure

$$\begin{aligned}
\mathbf{t}^*(\zeta) \quad &: \ \mathcal{W}_{\alpha-s,s} \ \rightarrow \ \mathcal{W}_{\alpha-s,s}\,,\\
\mathbf{b}^*(\zeta), \mathbf{c}(\zeta) \ &: \ \mathcal{W}_{\alpha-s+1,s-1} \ \rightarrow \ \mathcal{W}_{\alpha-s,s}\,,\\
\mathbf{c}^*(\zeta), \mathbf{b}(\zeta) \ &: \ \mathcal{W}_{\alpha-s-1,s+1} \ \rightarrow \ \mathcal{W}_{\alpha-s,s}\,.
\end{aligned}$$

$\mathbf{t}^*(\zeta)$ is bosonic and generates commuting integrals of motion. It commutes with all fermionic operators $\mathbf{b}(\zeta), \mathbf{c}(\zeta)$ and $\mathbf{b}^*(\zeta), \mathbf{c}^*(\zeta)$.

Anti-commutation relations and fermionic basis

Fermionic operators satisfy canonical anti-commutation relations

$$\big[\mathbf{c}(\zeta),\mathbf{c}^*(\zeta')\big]_+=\psi(\zeta/\zeta',\alpha),\quad \big[\mathbf{b}(\zeta),\mathbf{b}^*(\zeta')\big]_+=-\psi(\zeta'/\zeta,\alpha)$$

$$\text{with}\quad \psi(\zeta,\alpha)=\frac{1}{2}\,\zeta^{\alpha}\,\frac{\zeta^2+1}{\zeta^2-1}.$$

The operators **b** and **c** annihilate the “lattice primary field” $q^{2\alpha S(0)}$

$$\mathbf{b}(\zeta)\big(q^{2\alpha S(0)}\big)=0,\quad \mathbf{c}(\zeta)\big(q^{2\alpha S(0)}\big)=0,\quad S(k)=\frac{1}{2}\sum_{j=-\infty}^{k}\sigma_j^z.$$

Space of states is generated via multiple action of $\mathbf{t}^*(\zeta),\mathbf{b}^*(\zeta),\mathbf{c}^*(\zeta)$ on “primary field” $q^{2\alpha S(0)}$. In this way we get fermionic basis.

Mode expansions and locality

Annihilation operators are singular at $\zeta^2 \to 1$ while creation operators are regular:

$$\mathbf{b}(\zeta) = \zeta^{-\alpha-\mathbb{S}} \sum_{p=0}^{\infty} (\zeta^2 - 1)^{-p} \mathbf{b}_p, \quad \mathbf{c}(\zeta) = \zeta^{\alpha+\mathbb{S}} \sum_{p=0}^{\infty} (\zeta^2 - 1)^{-p} \mathbf{c}_p$$

$$\mathbf{b}^*(\zeta) = \zeta^{2+\alpha+\mathbb{S}} \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} \mathbf{b}^*_p, \quad \mathbf{c}^*(\zeta) = \zeta^{2-\alpha-\mathbb{S}} \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} \mathbf{c}^*_p$$

$$\mathbf{t}^*(\zeta) = \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} \mathbf{t}^*_p$$

Locality:

$$\mathbf{b}_p(X) = \mathbf{c}_p(X) = 0 \quad \text{for} \quad p > \mathrm{length}(X)$$

$$\mathrm{length}(\mathbf{b}^*_p(X)) \leq \mathrm{length}(X) + p, \quad \mathrm{length}(\mathbf{c}^*_p(X)) \leq \mathrm{length}(X) + p$$

$$\mathrm{length}(\mathbf{t}^*_p(X)) \leq \mathrm{length}(X) + p$$

Relation to the correlation functions

- Correlation functions of quasi-local operators $\mathcal{O}$ are generated by two transcendental functions ρ and ω. ρ is related to one-point function, ω is related to nearest neighbor correlators

$$\omega(\zeta,\zeta') = Z_{\mathbf{n}}\big(\mathbf{b}^*(\zeta)\mathbf{c}^*(\zeta')q^{2\alpha S(0)}\big)$$

  Both functions depend on temperature, disorder parameter and magnetic field, we call them **physical part**.

- In contrast to this, the basis is pure algebraic. It is built using representation theory of quantum group. We call it **algebraic part**.

The JMS-theorem allows to explicitly calculate Jimbo, Miwa, Smirnov (09)

$$Z_{\mathbf{n}}\Big\{\mathbf{t}^*(\zeta_1^0)\cdots\mathbf{t}^*(\zeta_p^0)\mathbf{b}^*(\zeta_1^+)\cdots\mathbf{b}^*(\zeta_q^+)\mathbf{c}^*(\zeta_q^-)\cdots\mathbf{c}^*(\zeta_1^-)\big(q^{\alpha\sum\limits_{j=-\infty}^{0}\sigma_j^z}\big)\Big\} =$$

$$= \prod_{i=1}^{p} 2\rho(\zeta_i^0)\det\Big|\omega(\zeta_i^+,\zeta_j^-)\Big|_{i,j=1,\cdots q}$$ generating function for series in ζ^2-1

The function ω

In this talk I mostly discuss the properties of the function ω.
In more general case:

$$\omega(\zeta,\zeta') = \omega(\zeta,\zeta'|\varkappa,\varkappa';\alpha), \quad \varkappa = (\kappa, S), \quad \varkappa' = (\kappa', S')$$

with different spin sectors S, S'

There are several equivalent definitions of ω

- via deformed Abelian integrals Jimbo, Miwa, Smirnov (09)
- via solution of linear and non-linear integral equations that come from thermodynamical description of 6-vertex model Göhmann, HB (09-12)
- via the function Φ. Göhmann, HB (12)

The function ω via integral equations

$$\frac{1}{4}\omega(\zeta,\xi|\varkappa,\varkappa';\alpha)=\frac{1}{4}\tilde{\omega}(\zeta,\xi|\varkappa,\varkappa';\alpha)+\omega_0(\zeta,\xi|\kappa,\kappa';\alpha),$$
$$\frac{1}{4}\tilde{\omega}(\zeta,\xi|\varkappa,\varkappa';\alpha)=-(f_{\mathrm{left}}\star G_{\mathrm{right}})(\zeta,\xi),\quad \omega_0(\zeta,\xi)=\delta^-_\zeta\delta^-_\xi\Delta^{-1}_\zeta\psi(\zeta/\xi,\alpha)$$

$$G_{\mathrm{right}}+K_\alpha\star G_{\mathrm{right}}=f_{\mathrm{right}},\quad (F\star G)(\zeta,\xi)=\int_\gamma dm(\eta)F(\zeta,\eta)\,G(\eta,\xi)$$

$$dm(\eta):=\frac{d\eta^2}{\eta^2\rho(\eta)(\mathfrak{a}(\eta)+1)},\quad \mathfrak{a}(\eta):=\frac{a(\eta)}{d(\eta)}\frac{Q(q\eta)}{Q(q^{-1}\eta)},\quad \rho(\eta):=\frac{T(\eta,\varkappa')}{T(\eta,\varkappa)}$$

$$\text{BAE:}\quad \mathfrak{a}(\lambda_i)=-1,\quad Q(\zeta):=Q(\zeta,\varkappa)=\zeta^{-\kappa+S}\prod_{i=1}^{\mathbf{n}/2-S}\left(1-\frac{\zeta^2}{\lambda_i^2(\varkappa)}\right)$$

integration contour γ goes around the points ζ^2,ξ^2 and all Bethe roots λ_i^2 in counterclockwise direction. Also we defined:

$$K_\alpha(\zeta,\xi):=\frac{1}{2\pi i}\Delta_\zeta\psi(\zeta/\xi,\alpha)\quad f_{\mathrm{left}}(\zeta,\xi):=\frac{1}{2\pi i}\delta^-_\zeta\psi(\zeta/\xi,\alpha),\quad f_{\mathrm{right}}(\zeta,\xi):=\delta^-_\xi\psi(\zeta/\xi,\alpha)$$
$$\Delta_\zeta f(\zeta):=f(q\zeta)-f(q^{-1}\zeta),\quad \delta^-_\zeta f(\zeta):=f(q\zeta)-\rho(\zeta)f(\zeta)$$

Define the function ω_+ replacing ψ by ψ_+ in all above definitions

$$\omega_+(\zeta,\xi) := \omega(\zeta,\xi)_{|_{\psi\to\psi_+}}, \quad \psi_+(\zeta,\alpha) = \frac{\zeta^\alpha}{\zeta^2-1}, \quad \psi(\zeta,\alpha) = \psi_+(\zeta,\alpha) + \frac{\zeta^\alpha}{2}$$

We can rewrite the functions $\tilde{\omega}$ and $\tilde{\omega}_+$ by taking ζ^2, ξ^2 out of the contour γ

$$\frac{1}{4}\tilde{\omega}(\zeta,\xi) = W(\zeta,\xi) + \sum_{i,j} V(\lambda_i,\zeta|-\alpha)(U^{-1})_{i,j} V(\lambda_j,\xi|\alpha),$$

$$W(\zeta,\xi) = \frac{\psi(q\zeta/\xi,\alpha)}{(1+\bar{\mathfrak{a}}(\zeta))(1+\mathfrak{a}(\xi))} - \frac{\psi(q^{-1}\zeta/\xi,\alpha)}{(1+\mathfrak{a}(\zeta))(1+\bar{\mathfrak{a}}(\xi))} - \left(\frac{\rho(\zeta)}{1+\mathfrak{a}(\xi)} - \frac{\rho(\xi)}{1+\mathfrak{a}(\zeta)}\right)\psi(\zeta/\xi,\alpha),$$

$$V(\zeta,\xi|\alpha) = \frac{\psi(q\zeta/\xi,\alpha)}{1+\mathfrak{a}(\xi)} + \frac{\psi(q^{-1}\zeta/\xi,\alpha)}{1+\bar{\mathfrak{a}}(\xi)} - \rho(\xi)\psi(\zeta/\xi,\alpha)$$

$$U_{i,j} = \frac{\delta_{i,j}}{m_i} + \psi(q\lambda_i/\lambda_j,\alpha) - \psi(q^{-1}\lambda_i/\lambda_j,\alpha), \quad m_i = \mathrm{res}_{\eta^2=\lambda_i^2}\ dm(\eta), \quad \bar{\mathfrak{a}}(\eta) := \frac{1}{\mathfrak{a}(\eta)}$$

and similar for $\tilde{\omega}_+$ by replacement $W \to W_+, V \to V_+, U \to U_+$

$$\frac{1}{4}\tilde{\omega}_+(\zeta,\xi) = W_+(\zeta,\xi) + \sum_{i,j} V_+(\lambda_i,\zeta|2-\alpha)(U_+^{-1})_{i,j} V_+(\lambda_j,\xi|\alpha)$$

Symmetries $\alpha \to -\alpha$ and $\alpha \to 2-\alpha$

Since we have

$$\psi(\zeta,\alpha) = -\psi(\zeta^{-1},-\alpha), \quad \psi_+(\zeta,\alpha) = -\psi_+(\zeta^{-1},2-\alpha)$$

we can check the following symmetries:

$$\omega(\zeta,\xi|\varkappa,\varkappa',\alpha) = \omega(\xi,\zeta|\varkappa,\varkappa',-\alpha),$$
$$\omega_+(\zeta,\xi|\varkappa,\varkappa',\alpha) = \omega_+(\xi,\zeta|\varkappa,\varkappa',2-\alpha)$$

As we will see both symmetries are important.

Proposition

$$\omega(\zeta,\xi|\varkappa,\varkappa',\alpha) = \omega_+(\zeta,\xi|\varkappa,\varkappa',\alpha)$$

if $\alpha = \kappa' - S' - \kappa + S, S' \geq S$ or $\alpha = -\kappa' + S' + \kappa - S, S' \leq S$
It seems to be more convenient to show this statement using the third 'Φ-formulation'.

The function ω via the function Φ

Introduce function $\Phi(\zeta,\xi) := \Phi(\zeta,\xi|\varkappa,\varkappa';\alpha)$

$$\Phi(\zeta,\xi) = \tilde{\Phi}(\zeta,\xi) + \Delta_\zeta^{-1}\psi(\zeta/\xi,\alpha),$$

where the following relation should be satisfied

$$\mathrm{res}_{\zeta^2=\lambda_i^2}\left(\frac{1}{\rho(\zeta)(1+\mathfrak{a}(\zeta))} + \frac{\tilde{\Phi}(\zeta,\xi)}{\Delta_\zeta\Phi(\zeta,\xi)}\right) = 0$$

It can be checked that $\quad \tilde{\Phi}(\zeta,\xi) = \sum_{i,j}\psi(\zeta/\lambda_i,\alpha)\,(U^{-1})_{i,j}\,\psi(\xi/\lambda_j,-\alpha)$

solves the above relation. Then the functions ω and ω_+ look:

$$\frac{1}{4}\omega(\zeta,\xi) = H_\zeta\; H_\xi\;\; \Phi(\zeta,\xi), \quad (H_\zeta f)(\zeta) := \frac{1}{1+\bar{\mathfrak{a}}(\zeta)}f(q\zeta) + \frac{1}{1+\mathfrak{a}(\zeta)}f(q^{-1}\zeta) - \rho(\zeta)f(\zeta)$$

$$\frac{1}{4}\omega_+(\zeta,\xi) = H_\zeta\; H_\xi\;\; \Phi_+(\zeta,\xi), \quad \Phi_+(\zeta,\xi) = \tilde{\Phi}_+(\zeta,\xi) + \Delta_\zeta^{-1}\psi_+(\zeta/\xi,\alpha)$$

$$\tilde{\Phi}_+(\zeta,\xi) = \sum_{i,j}\psi_+(\zeta/\lambda_i,\alpha)\,(U_+^{-1})_{i,j}\,\psi_+(\xi/\lambda_j,2-\alpha)$$

Further properties

Using the above results, we can show

Proposition: **Normalization condition**

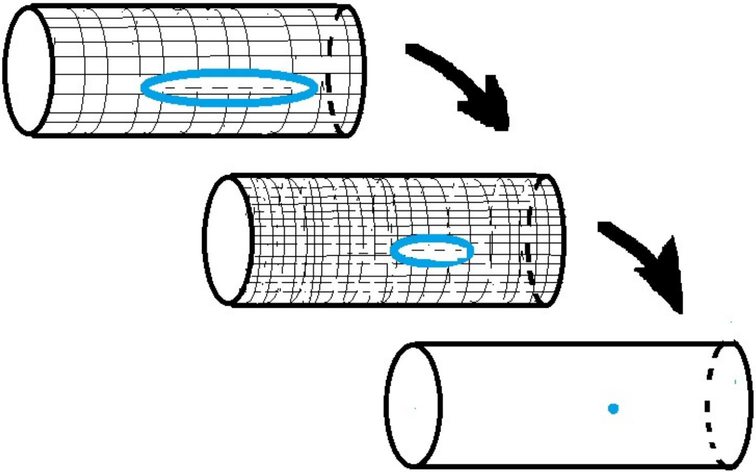
If the above relations of α to κ and κ' is satisfied and $0 < \alpha < 2$ then

$$\lim_{\zeta\to\infty} \omega_+(\zeta,\xi) = \lim_{\zeta\to 0} \omega_+(\zeta,\xi) = \lim_{\xi\to\infty} \omega_+(\zeta,\xi) = \lim_{\xi\to 0} \omega_+(\zeta,\xi) = 0$$

Proposition: $\varkappa \leftrightarrow \varkappa'$ **symmetry** Smirnov, HB (23)

$$\omega_+(\zeta,\xi|\varkappa,\varkappa';\alpha) = \rho(\zeta|\varkappa,\varkappa')\,\rho(\xi|\varkappa,\varkappa')\,\omega_+(\zeta,\xi|\varkappa',\varkappa;\alpha)$$

Application to QFT: continuum limit



Scaling limit

Aim is two-fold :

- to obtain the CFT with non-trivial $c = 1 - 6\nu^2/(1-\nu)$, $q = e^{\pi i \nu}$, $1/2 < \nu < 1$
- to consider asymptotic series for $\kappa \to \infty$

Scaling: Introduce lattice spacing a and take

$\tau_{\mathbf{m}} = q^{1/2}$, $\mathbf{n} \to \infty$, $a \to 0$, $\mathbf{n}a = 2\pi R$ with fixed radius of cylinder R

Lieb distribution gives: $\lambda_j \simeq (\pi j/\mathbf{n})^{\nu}$. Spectral parameter must be re-scaled:

$$\zeta = \lambda \bar{a}^{\nu},\ \bar{a} = Ca,\ C = \frac{\Gamma\left(\frac{1-\nu}{2\nu}\right)}{2\sqrt{\pi}\Gamma\left(\frac{1}{2\nu}\right)}\Gamma(\nu)^{\frac{1}{\nu}}$$

$$\rho^{\mathrm{sc}}(\lambda) = \lim_{\mathrm{scaling}} \rho(\lambda\bar{a}^{\nu}), \quad \omega^{\mathrm{sc}}(\lambda,\mu) = \frac{1}{4}\lim_{\mathrm{scaling}} \omega(\lambda\bar{a}^{\nu},\mu\bar{a}^{\nu})$$

Conjectures on operators in scaling:

- The creation operators are well-defined in the scaling limit for space direction when $ja = x$ is finite

$$\tau^*(\lambda) = \lim_{a\to 0}\frac{1}{2}\mathbf{t}^*(\lambda\bar{a}^{\nu}), \quad \beta^*(\lambda) = \lim_{a\to 0}\frac{1}{2}\mathbf{b}^*(\lambda\bar{a}^{\nu}), \quad \gamma^*(\lambda) = \lim_{a\to 0}\frac{1}{2}\mathbf{c}^*(\lambda\bar{a}^{\nu})$$

Asymptotic expansions at $\lambda \to \infty$ look

$$\log\bigl(\tau^*(\lambda)\bigr) \simeq \sum_{j=1}^{\infty}\tau^*_{2j-1}\lambda^{-\frac{2j-1}{\nu}}$$

$$\frac{1}{\sqrt{\tau^*(\lambda)}}\beta^*(\lambda) \simeq \sum_{j=1}^{\infty}\beta^*_{2j-1}\lambda^{-\frac{2j-1}{\nu}}, \quad \frac{1}{\sqrt{\tau^*(\lambda)}}\gamma^*(\lambda) \simeq \sum_{j=1}^{\infty}\gamma^*_{2j-1}\lambda^{-\frac{2j-1}{\nu}}.$$

Integrals of motion

- In 1987 Alexander Zamolodchikov introduced local integrals of motion which act on local operators as

$$(\mathbf{i}_{2n-1}O)(w)=\int_{C_w}\frac{dz}{2\pi i}h_{2n}(z)O(w)\qquad (n\geq 1)$$

where the densities $h_{2n}(z)$ are certain descendants of the identity operator I. An important property is that

$$\langle\Delta_-|\mathbf{i}_{2n-1}\big(O(z)\big)|\Delta_+\rangle=(I^+_{2n-1}-I^-_{2n-1})\langle\Delta_-|O(z)|\Delta_+\rangle$$
$$L_n|\Delta_+\rangle=\delta_{n,0}\Delta_+|\Delta_+\rangle\quad n\geq 0,\quad \langle\Delta_-|L_n=\delta_{n,0}\Delta_-\langle\Delta_-|\quad n\leq 0$$

where $I^{\pm}_{2n-1}$ denote the vacuum eigenvalues of the local integrals of motion on the Verma module with conformal dimension $\Delta_\pm$. The Verma module is spanned by the elements

$$\mathbf{i}_{2k_1-1}\cdots\mathbf{i}_{2k_p-1}\mathbf{l}_{-2l_1}\cdots\mathbf{l}_{-2l_q}(\Phi_\alpha(0)),\quad \Delta_\alpha=\frac{\nu^2\alpha(\alpha-2)}{4(1-\nu)}$$

In case when $\Delta_+=\Delta_-$ the space is spanned by the even Virasoro generators $\{\mathbf{l}_{-2n}\}_{n\geq 1}$.

Asymptotic expansions

- Using the result by Bazhanov, Lukyanov, Zamolodchikov (96-99), we get asymptotic expansion

$$\log \rho^{\mathrm{sc}}(\lambda) \simeq \sum_{j=1}^{\infty} \lambda^{-\frac{2j-1}{\nu}} C_j \left(I^{+}_{2j-1} - I^{-}_{2j-1} \right) \rightarrow \tau^{*}_{2j-1} = C_j \mathbf{i}_{2j-1}$$

$$\omega^{\mathrm{sc}}(\lambda,\mu) \simeq \sqrt{\rho^{\mathrm{sc}}(\lambda)\rho^{\mathrm{sc}}(\mu)} \sum_{i,j=1}^{\infty} \lambda^{-\frac{2i-1}{\nu}} \mu^{-\frac{2j-1}{\nu}} \omega_{i,j}$$

Scaling limit of the determinant formula

$$Z_R^{\kappa,\kappa'} \left\{ \tau^*(\lambda_1^0) \cdots \tau^*(\lambda_p^0) \beta^*(\lambda_1^+) \cdots \beta^*(\lambda_r^+) \gamma^*(\lambda_r^-) \cdots \gamma^*(\lambda_1^-) \big(\Phi_\alpha(0)\big) \right\}$$
$$= \prod_{i=1}^{p} \rho^{\mathrm{sc}}(\lambda_i^0) \times \det\left(\omega^{\mathrm{sc}}(\lambda_i^+, \lambda_j^-) \right)_{i,j=1,\dots,r}.$$

Technical problem: We get coefficients $\omega_{i,j}$ by the Wiener-Hopf technique only for $\kappa = \kappa'$ when $\Delta_+ = \Delta_-$ and $\rho^{\mathrm{sc}}(\zeta) = 1$ i.e. modulo the integrals of motion. $\Delta_+ = \frac{\nu^2}{4(1-\nu)}(\kappa^2 - 1), \quad \Delta_- = \frac{\nu^2}{4(1-\nu)}(\kappa'^2 - 1)$

Correspondence to CFT 3-point correlator

- It is possible to state the correspondence

$$\frac{\langle\Delta_-|P_\alpha\big(\{\mathbf{I}_{-k}\}\big)\Phi_\alpha(0)|\Delta_+\rangle}{\langle\Delta_-|\Phi_\alpha(0)|\Delta_+\rangle} = \lim_{\mathbf{n}\to\infty, a\to 0, \mathbf{n}a=2\pi R} Z_{\mathbf{n}}\big\{q^{2\alpha S(0)}\mathcal{O}\big\}$$

between a polynomial $P_\alpha(\{\mathbf{I}_{-k}\})$ and some combinations of $\beta^*_{2j-1}, \gamma^*_{2j-1}$.

Introduce $\beta^*_{2m-1} = D_{2m-1}(\alpha)\beta^{\mathrm{CFT}*}_{2m-1}, \quad \gamma^*_{2m-1} = D_{2m-1}(2-\alpha)\gamma^{\mathrm{CFT}*}_{2m-1}$

and take even and odd bilinear combinations

$$\phi^{\mathrm{even}}_{2m-1,2n-1} = (m+n-1)\frac{1}{2}\big(\beta^{\mathrm{CFT}*}_{2m-1}\gamma^{\mathrm{CFT}*}_{2n-1} + \beta^{\mathrm{CFT}*}_{2n-1}\gamma^{\mathrm{CFT}*}_{2m-1}\big),$$

$$\phi^{\mathrm{odd}}_{2m-1,2n-1} = d_\alpha^{-1}(m+n-1)\frac{1}{2}\big(\beta^{\mathrm{CFT}*}_{2n-1}\gamma^{\mathrm{CFT}*}_{2m-1} - \beta^{\mathrm{CFT}*}_{2m-1}\gamma^{\mathrm{CFT}*}_{2n-1}\big),$$

$$d_\alpha = \frac{\nu(\nu-2)}{\nu-1}(\alpha-1)$$

Identification with Virasoro Verma module

- If we accept an equivalence of the spaces spanned by

$$\mathbf{i}_{2k_1-1}\cdots\mathbf{i}_{2k_p-1}\mathbf{l}_{-2l_1}\cdots\mathbf{l}_{-2l_q}\big(\Phi_\alpha(0)\big) \qquad \text{and}$$

$$\mathbf{i}_{2k_1-1}\cdots\mathbf{i}_{2k_p-1}$$
$$\times\,\phi^{\mathrm{even}}_{2m_1-1,2n_1-1}\cdots\phi^{\mathrm{even}}_{2m_r-1,2n_r-1}\phi^{\mathrm{odd}}_{2\bar m_1-1,2\bar n_1-1}\phi^{\mathrm{odd}}_{2\bar m_{\bar r}-1,2\bar n_{\bar r}-1}\big(\Phi_\alpha(0)\big)$$

we can identify modulo integrals of motion ($\Delta\equiv\Delta_\alpha$)

Jimbo, Miwa, Smirnov, HB (10) (HGSIV), HB (11)

$$\phi^{\mathrm{even}}_{1,1}\cong\mathbf{l}_{-2},\qquad \phi^{\mathrm{even}}_{1,3}\cong\mathbf{l}^2_{-2}+\frac{2c-32}{9}\mathbf{l}_{-4},\qquad \phi^{\mathrm{odd}}_{1,3}\cong\frac{2}{3}\,\mathbf{l}_{-4}$$

$$\phi^{\mathrm{even}}_{1,5}\cong\mathbf{l}^3_{-2}+\frac{c+2-20\Delta+2c\Delta}{3(\Delta+2)}\mathbf{l}_{-4}\mathbf{l}_{-2}\;+\;\cdots\;\mathbf{l}_{-6}$$

$$\phi^{\mathrm{odd}}_{1,5}\cong\frac{2\Delta}{\Delta+2}\mathbf{l}_{-4}\mathbf{l}_{-2}+\frac{56-52\Delta-2c+4c\Delta}{5(\Delta+2)}\,\mathbf{l}_{-6}$$

$$\phi^{\mathrm{even}}_{3,3}\cong\mathbf{l}^3_{-2}+\frac{6+3c-76\Delta+4c\Delta}{6(\Delta+2)}\mathbf{l}_{-4}\mathbf{l}_{-2}\;+\;\cdots\;\mathbf{l}_{-6}$$

Functions Ψ and Θ

BLZ (97) introduced function Ψ that we also used in HGSIV-paper together with function Θ. Function Ψ is related to the integrals of motion:

$$I_{2n-1} = -i\,\Psi\Big(\frac{i(2n-1)}{2\nu},\kappa\Big)\; n(2n-1)(2\nu^2)^{n-1}(f\kappa)^{2n-1}R^{-2n+1}$$

$$\omega^{\rm sc}(\lambda,\mu) \simeq \sum_{r,s=1}^{\infty} \lambda^{-\frac{2r-1}{\nu}}\mu^{-\frac{2s-1}{\nu}} D_{2r-1}(\alpha)D_{2s-1}(2-\alpha)\,\frac{1}{\nu}\left(\frac{\sqrt{2}f\kappa\nu}{R}\right)^{2r+2s-2}$$
$$\times\Theta\Big(\frac{i(2r-1)}{2\nu},\frac{i(2s-1)}{2\nu}\Big|\kappa,\alpha\Big) \quad \text{in case} \quad \kappa=\kappa', \quad \big(f^{-1}=2\sqrt{2(1-\nu)}\big)$$

In HGSIV we applied the Wiener-Hopf factorization technique which is a bit complicated.

Smirnov, Negro (13) used reflection relations based on the above symmetry of ω w.r.t. $\alpha\to-\alpha$ and $\alpha\to 2-\alpha$ to simplify computations.

Recent results

Recently we have tried to further simplify computations Adler, HB (23). We use renormalized Ψ (with $p = f\kappa$) and a functional $\mathcal{F}$:

$$\Psi_\nu(s,p) = \frac{1}{2\nu i}\Psi(\frac{s}{2\nu i},\frac{p}{2f\nu}),\quad S_\nu(s) = S(\frac{s}{2\nu i}),\quad S_\nu(s,\alpha) = S(\frac{s}{2\nu i},\alpha)$$

$$\mathcal{F}(s,p,\Phi_-) =$$

$$= \sum_{n\geq 0}\frac{p^{-n}}{n!}\,\mathrm{res}_{h_1}\cdots\mathrm{res}_{h_n}\,\Phi_-(h_1)\cdots\Phi_-(h_n)\Big(1+s-\sum_{j=1}^{n}h_j\Big)_n\,\zeta_0\big(n+s-\sum_{j=1}^{n}h_j,p\big)$$

with 'chiral' sources: $\Phi_+(h) := \sum_{n>0}\frac{a_n}{n}\,h^{n-1},\quad \Phi_-(h) := \sum_{n<0} a_n\,h^n,$

bosonic modes: $[a_n,a_m] = n\,\delta_{n,-m}$ and vacuum $|0>$: $a_n|0>=0$ for $n>0$.

$$\zeta_0(s,p) = \frac{p^s}{s}\zeta(s,p+1/2) = \frac{p}{s(s-1)} - \frac{1}{s}\sum_{m\geq 1}p^{-2m+1}(1-2^{-2m+1})\frac{B_{2m}}{(2m)!}(s)_{2m-1}$$

Proposition: The function Ψ fulfills the equation

$$\Psi_\nu(s,p) = \langle 0|\exp\Big\{\mathrm{res}_h\Big((S_\nu(h)-1)\Psi_\nu(h,p)\Phi_+(h)\Big)\Big\}\mathcal{F}(s,p,\Phi_-)|0\rangle$$

Function Θ

Proposition: The function $\Theta_\nu(s,s';p,\alpha):=\frac{1}{2\nu}\Theta\left(\frac{s}{2\nu i},\frac{s'}{2\nu i}\middle|\frac{p}{2f\nu},\alpha\right)$ fulfills equation

$$0=\mathrm{res}_h\,\mathrm{res}_{h'}\,\frac{S_\nu(h,2-\alpha)S_\nu(h',\alpha)}{s+h}\Theta_\nu(h',s';p,\alpha)\times$$
$$\langle 0|\exp\left\{\mathrm{res}_{h''}\left(\left(S_\nu(h'')-1\right)\Psi_\nu(h'',p)\Phi_+(h'')\right)\right\}\mathcal{G}(-h-h',p,\Phi_-)|0\rangle$$

where $\mathcal{G}(s,p,\Phi_-)$ is a functional of fields Φ_-:

$$\mathcal{G}(s,p,\Phi_-)=$$
$$=\sum_{n\geq 0}\frac{p^{-n-1}}{n!}\mathrm{res}_{h_1}\ldots\mathrm{res}_{h_n}\Phi_-(h_1)\ldots\Phi_-(h_n)\left(1+s-\sum h_j\right)_{n+1}\zeta_0\left(n+1+s-\sum h_j,p\right)$$

Remark: To get formulae for excited states with L_+ particles and L_- holes defined by positions $I_r^{(\pm)}, r=1,\cdots,L_\pm$, we replace in the above formulas the function ζ_0 by

$$\zeta_0(s,p)\to\zeta_0(s,p)+E(s,p)$$
$$E(s,p)=\frac{1}{s}\sum_{r=1}^{L_+}\left(1-\frac{I_r^{(+)}}{2p}\right)^{-s}-\frac{1}{s}\sum_{r=1}^{L_-}\left(1+\frac{I_r^{(-)}}{2p}\right)^{-s}.$$

For a single particle-hole excitation $L_+=L_-=1$ we reproduce the result of HB(11).

Examples of expansion coefficients

$$\Psi_\nu(s,p)=\sum_{n\geq 0}\Psi_\nu^{(n)}(s)p^{1-n},\quad \Theta_\nu(s,s';p,\alpha)=\sum_{n\geq 0}\Theta_\nu^{(n)}(s,s';\alpha)p^{-n}$$

$$\Psi_\nu^{(0)}(s)=-\frac{i}{s(s+i/(2\nu))},\quad \Psi_\nu^{(1)}(s)=0,\quad \Psi_\nu^{(2)}(s)=\frac{i}{24}(-1+12(m_0+m_1)),$$

$$\Psi_\nu^{(3)}(s)=-\frac{s-i/(2\nu)}{8}(m_0^2-m_1^2),$$

$$\Psi_\nu^{(4)}(s)=-\frac{s-i/(2\nu)}{2880\,\nu(1-\nu)}(i\,s\,\nu(1-\nu)(7/2+60(m_0^3+m_1^3))$$

$$+5(2\nu^2+11(1-\nu)(m_0^3+m_1^3)+5(1+\nu)(2\nu-1)(m_0+m_1)(6(m_0+m_1)-1)+\nu^2+3(1-\nu))$$

$$\Theta_\nu^{(0)}(s,s';\alpha)=-\frac{i}{s+s'},\quad \Theta_\nu^{(1)}(s,s';\alpha)=0,$$

$$\Theta_\nu^{(2)}(s,s';\alpha)=-\frac{\nu^2\alpha(2-\alpha)+2i(1-\nu)(-i+2\nu(s+s'))}{96\nu(1-\nu)}(1-12(m_0+m_1)),$$

$$\Theta_\nu^{(3)}(s,s';\alpha)=-\frac{m_0^2-m_1^2}{512\,\nu^2(1-\nu)^2}(-3\nu^4\alpha^2(2-\alpha^2)-24\,i\,\alpha(2-\alpha)(1-\nu)\nu^2(-i+\nu(s+s'))$$

$$+32(1-\nu)^2(-i+\nu(s+s'))(-i+2\nu(s+s')))$$

Freedom in definition of the creation operators

- "Gauge transform":

$$\mathbf{b}^* \to e^{-\Omega}\, \mathbf{b}^*\, e^{\Omega}, \quad \mathbf{c}^* \to e^{-\Omega}\, \mathbf{c}^*\, e^{\Omega}$$

$$\Omega = \frac{1}{(2\pi i)^2} \oint\limits_{\Gamma} \frac{d\zeta^2}{\zeta^2} \oint\limits_{\Gamma} \frac{d\xi^2}{\xi^2} \omega'(\zeta,\xi)\, \mathbf{c}(\xi)\mathbf{b}(\zeta), \quad \omega \to \omega + \omega'$$

- If we take $\omega' = 0$, we get for naive limit $\mathbf{n} \to \infty$ (without rescaling of Bethe roots):

$$Z_\infty(\mathcal{O} q^{\alpha S(0)}) = \frac{\langle \mathrm{vac}|\mathcal{O} q^{\alpha S(0)}|\mathrm{vac}\rangle}{\langle \mathrm{vac}|q^{\alpha S(0)}|\mathrm{vac}\rangle} = \begin{cases} 1, & \text{if } \mathcal{O} = \tau^m,\ m \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$$

For target space $\quad \mathcal{W}_{\alpha,0}: \quad \zeta^{-\alpha}\mathbf{b}^*(\zeta) \to 0, \quad \zeta^{\alpha}\mathbf{c}^*(\zeta) \to 0, \quad \zeta \to 0$

Screening operators

- The other "gauge" choice with $\omega' = -\omega_0$:
  $\mathbf{b}^* \to \mathbf{b}^*_0 = O(\zeta^{\alpha}), \quad \mathbf{c}^* \to \mathbf{c}^*_0 = O(\zeta^{2-\alpha}), \quad \zeta \to 0$

- Acting in the subspace $\mathcal{W}_{\alpha,0}$

$$\mathbf{b}^*_0(\zeta) = \sum_{j=1}^{\infty} \zeta^{\alpha-2+2j}\mathbf{b}^*_{\text{screen},j}, \quad \mathbf{c}^*_0(\zeta) = \sum_{j=1}^{\infty} \zeta^{-\alpha+2j}\mathbf{c}^*_{\text{screen},j}$$

  $\mathbf{b}^*_{\text{screen},j}, \mathbf{c}^*_{\text{screen},j}$ are non-local.

- Scaling

$$\beta^*_{\text{screen}}(\lambda) = \lim_{a\to 0} \frac{1}{2}\mathbf{b}^*_0(\lambda \bar{a}^{\nu}), \quad \gamma^*_{\text{screen}}(\lambda) = \lim_{a\to 0} \frac{1}{2}\mathbf{c}^*_0(\lambda \bar{a}^{\nu}) \quad \text{for} \quad \lambda \to 0$$

  and for $\rho = 1$

$$\beta^*_{\text{screen}}(\lambda) \simeq \sum_{j=1}^{\infty} \lambda^{\alpha+2j-2}\beta^*_{\text{screen},j}, \quad \gamma^*_{\text{screen}}(\lambda) \simeq \sum_{j=1}^{\infty} \lambda^{-\alpha+2j}\gamma^*_{\text{screen},j}$$

Fermionic construction of primary field

Let $\mathcal{V}_\alpha$ be the subspace obtained by acting $\beta^*_{2j-1},\gamma^*_{2j-1}$ and integrals of motion $\mathbf{i}_{2k-1}$. In case $\kappa=\kappa'$ we factor out the integrals of motion

$$\mathcal{V}^{\mathrm{quo}}_\alpha=\mathcal{V}_\alpha/\sum \mathbf{i}_{2k-1}\mathcal{V}_\alpha$$

The basis of $\mathcal{V}^{\mathrm{quo}}_\alpha$: $\quad \beta^*_{I^+},\gamma^*_{I^-}\Phi_\alpha(0)$ **moving inside Verma module**

$$\beta^*_{I^+}=\beta^*_{2k_1-1}\cdots\beta^*_{2k_n-1},\quad \gamma^*_{I^-}=\gamma^*_{2j_n-1}\cdots\gamma^*_{2j_1-1}.$$

Acting on $\Phi_\alpha(0)$ by $\beta^*_{2j-1},\gamma^*_{2j-1},\beta^*_{\mathrm{screen},j},\gamma^*_{\mathrm{screen},j}$, one gets a space $\mathcal{H}_\alpha\supset\mathcal{V}^{\mathrm{quo}}_\alpha$

The claim is: $\quad \mathcal{V}^{\mathrm{quo}}_{\alpha+2m\frac{(1-\nu)}{\nu}}\subset\mathcal{H}_\alpha,\quad m\in\mathbb{Z}_{\geq 0}$ Jimbo, Miwa, Smirnov (11)

Moving between Verma modules: $\quad \Phi_{\alpha+2m\frac{(1-\nu)}{\nu}}(0)\cong\beta^*_{I_{\mathrm{odd(m)}}}\gamma^*_{\mathrm{screen},I(m)}\Phi_\alpha(0)$

$$\gamma^*_{\mathrm{screen},I(m)}=\gamma^*_{\mathrm{screen},m}\cdots\gamma^*_{\mathrm{screen},1},\quad I(m)=(1,2,\cdots,m),\,I_{\mathrm{odd(m)}}=(1,3,\cdots,2m-1)$$

Conformal dimensions of operators:

$$\beta^*_{2j-1},\gamma^*_{2j-1}:\quad 2j-1,\quad \beta^*_{\mathrm{screen},j}:\quad \nu(2-\alpha-2j),\quad \gamma^*_{\mathrm{screen},j}:\quad \nu(\alpha-2j)$$

The case of the sine-Gordon model (sG)

The sG model: $S^{sG} = \int d^2x \left(\frac{1}{16\pi} (\partial_\mu \varphi(x))^2 + \frac{\mu}{\sin \pi\beta^2} 2\cos(\beta\varphi(x)) \right)$

$$\Phi_\alpha = e^{\frac{\nu\alpha}{2(1-\nu)}(i\beta\varphi)}, \quad \nu = 1-\beta^2, \quad p = \frac{\beta^2}{1-\beta^2}, \quad \frac{1}{2} < \nu < 1, \quad 0 < \alpha < 2$$

is considered as integrable perturbation of the CFT. Also it can be obtained as continuum limit of the six-vertex model. The homogeneous lattice should be replaced by the inhomogeneous one: $\zeta_j = \zeta_0^{(-1)^j}, \tau_{\mathbf{m}} = \zeta_0^{(-1)^{\mathbf{m}}}$

Scaling limit: $\mathbf{n} \to \infty, \quad a \to 0, \quad \mathbf{n}a = 2\pi R, \quad \zeta_0 \to \infty, \quad \mu = \zeta_0^{-1}(Ca)^{-\nu}$

Claim: The fermionic basis can be applied to sG model Jimbo, Miwa, Smirnov (10-11)

The creation operators $\mathbf{b}^*, \mathbf{c}^*$ split into the pair of operators $\mathbf{b}^{\pm *}, \mathbf{c}^{\pm *}$ stemming from the expansion series around $\zeta^2 = \zeta_0^{\pm 2}$ respectively with the nearest-neighbor correlators:

$$Z_{\mathbf{n}}\{\mathbf{b}^{\pm *}(\zeta)\mathbf{c}^{\pm *}(\xi)(q^{2\alpha S(0)})\} = \omega(\zeta,\xi),$$
$$Z_{\mathbf{n}}\{\mathbf{b}^{\pm *}(\zeta)\mathbf{c}^{\mp *}(\xi)(q^{2\alpha S(0)})\} = \tilde{\omega}(\zeta,\xi)$$

The space of the descendants of the exponential field Φ_α in the sG model is identified with the tensor product of Verma modules $\mathcal{V}_\alpha \otimes \bar{\mathcal{V}}_\alpha$. Considering CFT and sG model as scaling limits of homogeneous and inhomogeneous six-vertex models, respectively, one can expect that the action of local integrals of motion $\mathbf{i}_{2j-1}, \bar{\mathbf{i}}_{2j-1}$ and fermions $\beta^*_{2j-1}, \gamma^*_{2j-1}, \bar{\beta}^*_{2j-1}, \bar{\gamma}^*_{2j-1}$ for CFT are in one-to-one correspondence with those for sG. Also one can consider $\mathcal{V}^{\text{quo}}_\alpha \otimes \bar{\mathcal{V}}^{\text{quo}}_\alpha$. Two chiralities emerge via scaling:

$$\frac{1}{2}\mathbf{b}^{+*} \xrightarrow[\text{scaling}]{} \beta^{+*}(\zeta) \xrightarrow[\zeta\to\infty]{} \beta^*_{2j-1}, \bar{\beta}^*_{\text{screen},j}, \quad \frac{1}{2}\mathbf{c}^{+*} \xrightarrow[\text{scaling}]{} \gamma^{+*}(\zeta) \xrightarrow[\zeta\to\infty]{} \gamma^*_{2j-1}, \bar{\gamma}^*_{\text{screen},j}$$

$$\frac{1}{2}\mathbf{b}^{-*} \xrightarrow[\text{scaling}]{} \beta^{-*}(\zeta) \xrightarrow[\zeta\to 0]{} \bar{\beta}^*_{2j-1}, \beta^*_{\text{screen},j}, \quad \frac{1}{2}\mathbf{c}^{-*} \xrightarrow[\text{scaling}]{} \gamma^{-*}(\zeta) \xrightarrow[\zeta\to 0]{} \bar{\gamma}^*_{2j-1}, \gamma^*_{\text{screen},j}$$

The identification of the shifted primary field can be got from that of the CFT:

$$\Phi_{\alpha+2pm}(0) \cong C_m(\alpha)\, \beta^*_{I_{\text{odd}}(m)}\, \bar{\gamma}^*_{I_{\text{odd}}(m)}\, \Phi^{(m)}_\alpha(0)$$

$$\Phi^{(m)}_\alpha(0) = i^m \mu^{2m} \prod_{j=0}^{m-1} \operatorname{ctg}(\pi\nu(j-\alpha/2))\ \bar{\beta}^*_{\text{screen},\mathrm{I(m)}}\, \gamma^*_{\text{screen},\mathrm{I(m)}}\, \Phi_\alpha(0)$$

In particular, $\quad \Phi_{\alpha+2p}(0) \cong -i\mu^2 \operatorname{ctg}(\pi\nu\alpha/2) C_1(\alpha)\, \beta^*_1\, \bar{\gamma}^*_1\, \bar{\beta}^*_{\text{screen},1}\, \gamma^*_{\text{screen},1}\, \Phi_\alpha(0)$

Simplest examples

Some ratios can be computed and related to the function ω:

Jimbo, Miwa, Smirnov (10) (HGSV)

- for the CFT when $\Delta_+ = \Delta_-$

$$\frac{\langle\Delta_-|\Phi_{\alpha+2p}(0)|\Delta_+\rangle}{\langle\Delta_-|\Phi_{\alpha}(0)|\Delta_+\rangle} = -i\mu^2 \,\mathrm{ctg}\,(\pi\nu\alpha/2)C_1(\alpha)\, Z_\infty\{\beta_1^*\,\bar{\gamma}_1^*\,\bar{\beta}^*_{\mathrm{screen},1}\,\gamma^*_{\mathrm{screen},1}(q^{2\alpha S(0)})\}$$

 and compared with the Dorn-Otto-Zamolodchikov-Zamolodchikov formula.

- for one-point function of the sG model

$$\frac{\langle\Phi_{\alpha+2p}(0)\rangle^{\mathrm{sG}}_{R=\infty}}{\langle\Phi_{\alpha}(0)\rangle^{\mathrm{sG}}_{R=\infty}}$$

and compared with the Lukyanov-Zamolodchikov formula.

The function $\mathfrak{A}$

It is more convenient to change variables. Introduce $Z := \zeta^{p+1}, X := \xi^{p+1}$.

$$\mathcal{R}(Z) := \rho(Z^{\frac{1}{p+1}}), \quad \Psi(Z,\alpha) := \frac{2}{p+1}\psi_+(Z^{\frac{1}{p+1}},\alpha),$$

$$\mathcal{K}_\alpha(Z) := \frac{1}{2\pi i}\big(\Psi(e^{i\pi}Z,\alpha) - \Psi(e^{-i\pi}Z,\alpha)\big)$$

$$\Omega(Z,X|\alpha) := \frac{1}{4}\cdot\frac{2}{p+1}\,\omega(Z^{\frac{1}{p+1}},X^{\frac{1}{p+1}})$$

$$f_\alpha(Z,X) := \Psi(e^{-i\pi}Z/X,\alpha) - \mathcal{R}(X)\Psi(Z/X,\alpha)$$

To describe the function Ω, we start with the DDV-equation:

$$\frac{1}{i}\log\mathfrak{A}(Z|\kappa) = 2\pi MR(Z - Z^{-1}) - \frac{4\pi\kappa}{p} - 2\int_0^\infty \frac{dS}{S}\, G(Z/S)\,\mathrm{Im}\log\Big(1 + \mathfrak{A}(Se^{+i0}|\kappa)\Big)$$

with soliton mass M related to μ: $\quad \mu = (\frac{1}{4}MC^{-1})^\nu$ and the kernel

$$G(Z) = \int_{-\infty}^{\infty} dk\, Z^{ik}\,\frac{\sinh\frac{\pi(1-2p)k}{2}}{4\pi\sinh(\frac{\pi pk}{2})\cosh(\frac{\pi k}{2})}$$

The function $\mathcal{R}$ and resolvents

$$\log \mathcal{R}(Z) = \frac{1}{\pi}\int_0^\infty \frac{dS}{S}\,\mathrm{Im}\left(\frac{2ZX}{Z^2e^{-i0}-S^2}\,L(Se^{i0})\right)$$

$$L(S) := \log\left(1+\mathfrak{A}(Z|\kappa')\right) - \log\left(1+\mathfrak{A}(Z|\kappa)\right)$$

Asymtotically: $\log \mathcal{R}(Z) \underset{Z\to\infty}{\simeq} \sum_{j=1}^{\infty} Z^{-2j+1}\Delta I_{2j-1},\ \log \mathcal{R}(Z) \underset{Z\to 0}{\simeq} \sum_{j=1}^{\infty} Z^{2j-1}\Delta \bar{I}_{2j-1}$

In the CFT limit: $\Delta I_{2j-1} = M^{-2j+1}C_{2j-1}\left(I_{2j-1}(\kappa') - I_{2j-1}(\kappa)\right)$

Dressed resolvent (two steps definition):

- 'bare' resolvent $\mathcal{R}_\alpha$:

  $\mathcal{R}_\alpha - \mathcal{K}_\alpha \circ \mathcal{R}_\alpha = \mathcal{K}_\alpha, \quad (F\circ F')(Z,X) := \int_0^\infty \frac{dS}{S\mathcal{R}(S)} F(Z,S)F'(S,X)$

- 'dressed' resolvent $\mathcal{R}_\alpha^{\mathrm{dress}}$: $\quad \mathcal{R}_\alpha^{\mathrm{dress}} + \mathcal{R}_\alpha * \mathcal{R}_\alpha^{\mathrm{dress}} = \mathcal{R}_\alpha$

$$(F*F')(Z,X) := \int_0^\infty \frac{dS}{S\mathcal{R}(S)}\mathrm{Re}\left(\frac{1}{1+\mathfrak{A}(Se^{-i0})}\right) F(Z,S)F'(S,X)$$

The function Ω

First, we introduce

$$F_\alpha := f_\alpha + \mathcal{R}^{\text{dress}}_\alpha * f_\alpha$$

The function Ω splits into two parts:

$$\Omega(Z,X|\alpha) = \Omega^{(1)}(Z,X|\alpha) + \Omega^{(2)}(Z,X|\alpha)$$

$$\Omega^{(1)}(Z,X|\alpha) := -\frac{1}{2\pi i}\Big(F_{2-\alpha} * F_\alpha - F_{2-\alpha} * \mathcal{R}^{\text{dress}}_\alpha * F_\alpha\Big)(Z,X)$$

$$\Omega^{(2)}(Z,X|\alpha) := U(Z,X) - \frac{1}{\pi i}(U \circ F_\alpha)(Z,X)$$

$$U(Z,X) := \frac{1}{4}\big(1-\mathcal{R}(Z)\big)\big(1+\mathcal{R}(X)\big)\frac{Z^2+X^2}{Z^2-X^2} - \frac{1}{4}\big(1+\mathcal{R}(Z)\big)\big(1-\mathcal{R}(X)\big)\frac{2\,Z\,X}{Z^2-X^2}$$

Asymptotics $(\varepsilon, \varepsilon' = \pm)$ (numerical computation of few coefficients Ω): Smirnov,HB (18)

$$\Omega(Z,X|\alpha) \underset{\substack{\log(Z)\to\varepsilon\cdot\infty \\ \log(X)\to\varepsilon'\cdot\infty}}{\simeq} \big(\mathcal{R}(Z)\mathcal{R}(X)\big)^{\frac{1}{2}} \sum_{j,k=1}^{\infty} Z^{-\varepsilon(2j-1)} X^{-\varepsilon'(2k-1)} \Omega_{\varepsilon(2j-1),\varepsilon'(2k-1)}(\alpha)$$

Shift relations

There should exist relation wrt shift of α. Asymptotics wrt the first argument:

$$\Omega(Z,X|\alpha) \underset{\log(Z)\to\pm\infty}{\simeq} \left(\mathcal{R}(Z)\right)^{\frac{1}{2}} \sum_{j=1}^{\infty} Z^{\mp(2j-1)}\Omega_{\pm(2j-1),\star}(X|\alpha)$$

Theorem Smirnov, HB (23)

$$(X/Z)^2\,\Omega(Z,X|\alpha+2p)-\Omega(Z,X|\alpha)+\frac{\left(\Omega_{-1}(Z|2-\alpha)+\Xi^{\dagger}(Z|\alpha)\right)\left(\Omega_{1}(X|\alpha)+\Xi(X|\alpha)\right)}{\Omega_{1,-1}(\alpha)+\Xi(\alpha)}$$

$$=\Xi(Z,X|\alpha),$$

$$\Xi(Z,X|\alpha):=\frac{1}{2i}\{2t_1(\alpha)\frac{X}{Z}(1+\mathcal{R}(Z))(1+\mathcal{R}(X))$$

$$-\left(\left(\frac{X}{Z}\right)^2 t_2(\alpha)+t_0(\alpha)\right)(1-\mathcal{R}(Z))(1-\mathcal{R}(X))\}, \qquad t_j(\alpha):=\frac{1}{2}\operatorname{ctg}\frac{\pi}{2}(\alpha+j(p+1))$$

$$\Xi(Z,X|\alpha)\underset{Z\to\infty}{\simeq} Z^{-1}\,\Xi(X|\alpha),\quad \Xi(X|\alpha)=\frac{1}{2i}(4t_1(\alpha)X(1+\mathcal{R}(X))+t_0(\alpha)\Delta I_1(1-\mathcal{R}(X)))$$

$$\Xi(Z,X|\alpha)\underset{X\to 0}{\simeq} X\,\Xi^{\dagger}(Z|\alpha),\ \Xi^{\dagger}(Z|\alpha)=\frac{1}{2i}(4t_1(\alpha)Z^{-1}(1+\mathcal{R}(Z))+t_0(\alpha)\Delta\bar{I}_1(1-\mathcal{R}(Z)))$$

$$\Xi(\alpha):=\frac{1}{2i}(2t_1(\alpha)-\Delta I_1\Delta\bar{I}_1\,t_0(\alpha))$$

In some sense the above shift relation can be 'solved'. Consider three positive integers m, i, j such that $m \geq 1$, $i, j > m$. Then

$$\Omega_{2(i-m)-1,-(2(j-m)-1)}(\alpha+2mp) = \frac{\det\left(A^{(m,i,j)}(\alpha)\right)}{\det\left(A^{(m)}(\alpha)\right)}$$

where $A^{(m)}(\alpha)$ is a $m \times m$ matrix and $A^{(m,i,j)}(\alpha)$ is a $(m+1)\times(m+1)$ matrix with the elements determined for $1 \leq k, l \leq m$ as follows:

$$\left(A^{(m)}(\alpha)\right)_{k,l} = \left(A^{(m,i,j)}(\alpha)\right)_{k,l} = \Omega_{2k-1,-(2l-1)}(\alpha) + \Xi^{(m)}_{2k-1,-(2l-1)}(\alpha),$$

$$\left(A^{(m,i,j)}(\alpha)\right)_{k,m+1} = \Omega_{2k-1,-(2j-1)}(\alpha) + \Xi^{(m)}_{2k-1,-(2j-1)}(\alpha)$$

$$\left(A^{(m,i,j)}(\alpha)\right)_{m+1,l} = \Omega_{2i-1,-(2l-1)}(\alpha) + \Xi^{(m)}_{2i-1,-(2l-1)}(\alpha)$$

$$\left(A^{(m,i,j)}(\alpha)\right)_{m+1,m+1} = \Omega_{2i-1,-(2j-1)}(\alpha) + \Xi^{(m)}_{2i-1,-(2j-1)}(\alpha)$$

$$\Xi^{(m)}(Z,X|\alpha) := \sum_{r=0}^{m-1} \Xi(Z,X|\alpha+2pr)(X/Z)^{2r}$$

The above solution provides many identities stemming from relations between Verma modules with shifted α.

One example of such identity from HGSV-paper that follows from the above relation taken at $i=j=m+1$ (can be shown by induction):

$$\det\left(A^{(m,m+1,m+1)}(\alpha)\right)=\prod_{k=0}^{m}\left(A^{(k)}(\alpha+2(m-k)p)\right)_{1,1}$$

The claim is that such identities have to do with certain Grassmannians.

Conclusions

- We saw that the function ω has many symmetries, in particular, the $\kappa \leftrightarrow \kappa'$ symmetry which is satisfied inspite of the fact that these two parameters enter in rather different manner. Probably, there exists an explicitly symmetric description which might help us to solve what we call ρ-problem and escape the condition $\kappa = \kappa'$ in order to involve the contribution of the integrals of motion.
- Also the function ω in invariant under two reflections:
$$\sigma_1 : \alpha \to 2-\alpha, \quad \text{and} \quad \sigma_2 : \alpha \to -\alpha$$
where the first one is related to the natural symmetry of the CFT since $\Delta_\alpha = \Delta_{2-\alpha}$ while the second reflection originates from the sG model. The idea to use both these symmetries was promoted by Negro and Smirnov (13). Also it helped us together with Smirnov (18) to incorporate the integrals of motion for few particular cases of Virasoro levels.
- We expect that the last discussed α-shift relation for ω should be helpful in further understanding of the shifted Virasoro modules and, in particular, of different identities discovered by Jimbo, Miwa and Smirnov (11) in this context.

Since this is very special conference on the occasion of 65-th birthday of my dear friend and co-author, Fedya Smirnov, let me use this opportunity and wish Fedya good health and very many years of fruitful work in future!