Space-like asymptotics of the transverse two-point functions of the XXZ chain at finite temperature

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Outline

1. The thermal form factor expansion
   - The non-linear integral equation approach to the spectrum
   - The thermal form factor series
2. The large-m, $|t| < \mathfrak{c}m$, analysis at low-T
3. Conclusion

The non-linear integral equation

⊛ **Idea** ('91 **Klümper** , '92 **Destri-DeVega**):
solutions of BAE ⇝ Solutions of constrained Non-Linear Integral Equations

♦ Auxiliary function $\widehat{u}$:

$$\widehat{\mathfrak{a}}(\xi) = \mathrm{e}^{-\frac{1}{T}\widehat{u}(\xi)} = \mathrm{e}^{-\frac{h}{T}}(-1)^{\mathfrak{s}} \frac{d(\xi \mid N, T, t)}{a(\xi \mid N, T, t)} \prod_{k=1}^{N-\mathfrak{s}} \left\{ \frac{\sinh(\mathrm{i}\zeta - \xi + \lambda_k)}{\sinh(\mathrm{i}\zeta + \xi - \lambda_k)} \right\}$$

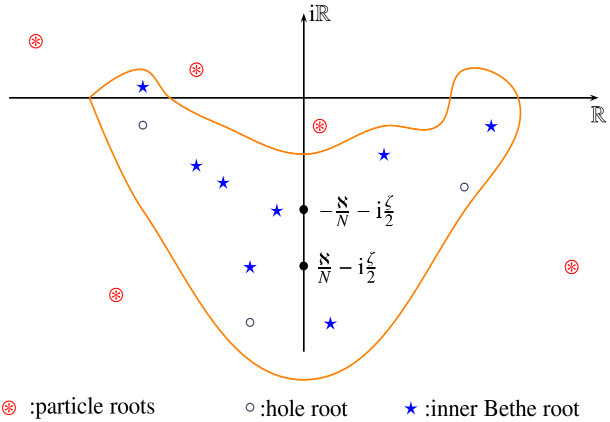
- $\widehat{\mathfrak{a}}(\lambda_a) = -1$ by construction BAE
- Alternative characterisation of $\widehat{u}$ ⇝ tool to characterise the thermodynamics

Non-Linear Integral Equation for $\widehat{u}$ set up by residue calculation

$$\widehat{u}(\xi) = \mathcal{D}_N(\xi) + \mathrm{i}\pi\mathfrak{s} - \mathrm{i}T\sum_{a=1}^{M}\theta(\xi - \lambda_a)$$

Setting up the NLIE

- Pick a domain $\mathcal{D} \subset \mathbb{C}$
- $\{\lambda_a\}_1^M \cap \mathcal{D}^c = \widehat{\mathcal{Y}} = \{\widehat{y}_a\}_1^{|\widehat{\mathcal{Y}}|}$ Bethe roots outside of $\mathcal{D}$ "particles" $\mathrm{e}^{-\frac{1}{T}\widehat{u}(\widehat{y}_a)} = -1$
- $\lambda \in \{\lambda_a\}_1^M \setminus \widehat{\mathcal{Y}}$ Bethe roots inside of $\mathcal{D}$ $\mathrm{e}^{-\frac{1}{T}\widehat{u}(\lambda)} = -1$
- $\widehat{\mathfrak{X}} = \{\widehat{x}_a\}_1^{|\widehat{\mathfrak{X}}|}$ non-Bethe roots inside of $\mathcal{D}$ "holes" $\mathrm{e}^{-\frac{1}{T}\widehat{u}(\widehat{x}_a)} = -1$



The *perse* NLIE

- Monodromy condition $$\oint_{\partial\mathcal{D}} \frac{\widehat{u}'(\eta)}{1+\mathrm{e}^{\frac{1}{T}\widehat{u}(\eta)}} \cdot \frac{\mathrm{d}\eta}{2\mathrm{i}\pi} = 0$$

- Non-linear integral equation

$$\widehat{u}(\xi \mid \boldsymbol{y}_m, \boldsymbol{x}_n) = h - T\mathfrak{w}_N(\xi) + \mathrm{i}\pi\mathfrak{s} - \mathrm{i}T\Theta(\xi \mid \boldsymbol{y}_m, \boldsymbol{x}_n) + \oint_{\partial\mathcal{D}} \theta'(\xi-\eta)\cdot\mathcal{L}\mathrm{n}\Big[1+\mathrm{e}^{-\frac{1}{T}\widehat{u}}\Big](\eta \mid \boldsymbol{y}_m, \boldsymbol{x}_n)\cdot\frac{\mathrm{d}\eta}{2\pi}$$

- ♦ Subdominant driving term

$$\Theta(\xi \mid \boldsymbol{y}_m, \boldsymbol{x}_n) = \sum_{a=1}^{m}\theta(\xi-y_a) - \sum_{a=1}^{n}\theta(\xi-x_a) \quad , \qquad \theta(\lambda) = \mathrm{i}\ln\left(\frac{\sinh(\mathrm{i}\zeta+\lambda)}{\sinh(\mathrm{i}\zeta-\lambda)}\right)$$

- Auxiliary conditions on $\widehat{\mathcal{Y}} = \big\{\widehat{\mathfrak{y}}_a\big\}_1^m$ and $\widehat{\mathfrak{X}} = \big\{\widehat{\mathfrak{x}}_a\big\}_1^n$

$$\mathrm{e}^{-\frac{1}{T}\widehat{u}(\widehat{\mathfrak{x}}_a \mid \widehat{\mathfrak{y}}_m, \widehat{\mathfrak{x}}_n)} = -1 \qquad\qquad \mathrm{e}^{-\frac{1}{T}\widehat{u}(\widehat{\mathfrak{y}}_a \mid \widehat{\mathfrak{y}}_m, \widehat{\mathfrak{x}}_n)} = -1$$

- The $\widehat{\mathfrak{a}}$ function: $$\widehat{\mathfrak{a}}(\xi) = \exp\Big\{-\tfrac{1}{T}\widehat{u}(\xi \mid \widehat{\mathfrak{y}}_m, \widehat{\mathfrak{x}}_n)\Big\}$$

Taking the Trotter limit

- Easy to take *formally* the $N \to +\infty$ limit
  - Pointwise Trotter limit of driving term

$$h - T\mathfrak{w}_N(\xi) \underset{N\to+\infty}{\longrightarrow} \varepsilon_0(\xi) = h - \frac{2J\sin^2(\zeta)}{\sinh(\xi + \mathrm{i}\frac{\zeta}{2})\sinh(\xi - \mathrm{i}\frac{\zeta}{2})}$$

  - Assuming $\widehat{\mathfrak{y}}_a, \widehat{\mathfrak{x}}_a, \widehat{u} \hookrightarrow \mathfrak{y}_a, \mathfrak{x}_a, u$, one has the limiting problem

$$u(\xi \mid \boldsymbol{y}_m, \boldsymbol{x}_n) = \varepsilon_0(\xi) + \mathrm{i}\pi\mathfrak{s} - \mathrm{i}T\Theta(\xi \mid \boldsymbol{y}_m, \boldsymbol{x}_n) + \oint_{\partial\mathcal{D}} \theta'(\xi - \eta) \cdot \mathcal{L}\mathrm{n}\Big[1 + \mathrm{e}^{-\frac{1}{T}u}\Big](\eta \mid \boldsymbol{y}_m, \boldsymbol{x}_n) \cdot \frac{\mathrm{d}\eta}{2\pi}$$

and

$$\mathrm{e}^{-\frac{1}{T}u(\mathfrak{x}_a \mid \mathfrak{y}_m, \mathfrak{x}_n)} = -1 \qquad\qquad \mathrm{e}^{-\frac{1}{T}u(\mathfrak{y}_a \mid \mathfrak{y}_m, \mathfrak{x}_n)} = -1$$

- Can be put on rigorous grounds for low T (’**23** **Faulmann, Göhmann, K.**)

Structure of solutions

- Control term for $N \gg 1$, $T \ll 1$ $\rightsquigarrow$ dressed energy in the complex plane:

  - $$\varepsilon_c(\lambda) + \int_{\mathscr{C}_\varepsilon} \theta'(\lambda-\mu)\varepsilon_c(\mu)\frac{d\mu}{2\pi} = \varepsilon_0(\lambda)\,,$$
  - $$\mathscr{C}_\varepsilon = \Big\{\lambda \,:\, -\frac{\pi}{2} \le \Im(\lambda) \le 0\,,\ \Re[\varepsilon_c(\lambda)] = 0\Big\}$$

Theorem '23 **Faulmann, Göhmann, K.**

Let $0 < \zeta < \dfrac{\pi}{2}$, $\exists T_0, \eta$ such that, for all $T_0 > T > 0$ and $\eta > \frac{1}{NT^4}$, it holds

$$\Re[\varepsilon_c(\widehat{\mathfrak{x}}_k)] = o(1) \quad \text{with} \quad \widehat{\mathfrak{x}}_k \in \mathcal{D}$$
$$\Re[\varepsilon_c(\widehat{\mathfrak{y}}_k)] = o(1) \quad \text{with} \quad \widehat{\mathfrak{y}}_k \in \Big\{z \,:\, -\tfrac{\pi}{2} < \Im(z) \le \tfrac{\pi}{2}\Big\} \setminus \mathcal{D}\,.$$

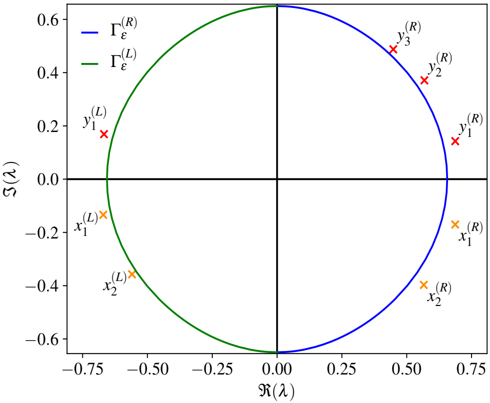
Given integers $0 \le p_1^{(\upsilon)} < \ldots < p^{(\upsilon)}_{|\mathcal{Y}^{(\upsilon)}|}$ *and* $0 \le h_1^{(\upsilon)} < \ldots < h^{(\upsilon)}_{|\mathfrak{X}^{(\upsilon)}|}$

there exists a unique solution $(\widehat{u}, \widehat{\mathfrak{X}}, \widehat{\mathcal{Y}})$ to the NLIE & auxiliary conditions such that

$$\widehat{u}(\widehat{\mathfrak{x}}_a^{(\upsilon)} \mid \widehat{\mathfrak{y}}_m, \widehat{\mathfrak{x}}_n) = \mp 2i\pi T(h_a^{(\upsilon)} + \tfrac{1}{2})$$
$$\widehat{u}(\widehat{\mathfrak{y}}_a^{(\upsilon)} \mid \widehat{\mathfrak{y}}_m, \widehat{\mathfrak{x}}_n) = \pm 2i\pi T(p_a^{(\upsilon)} + \tfrac{1}{2})$$

and

$$\widehat{\mathfrak{X}} = \bigcup_{\upsilon\in\{L,R\}} \big\{\widehat{x}_a^{(\upsilon)}\big\}_{a=1}^{|\mathfrak{X}^{(\upsilon)}|}$$
$$\widehat{\mathcal{Y}} = \bigcup_{\upsilon\in\{L,R\}} \big\{\widehat{y}_a^{(\upsilon)}\big\}_{a=1}^{|\mathcal{Y}^{(\upsilon)}|}$$



$p_3^{(R)} = 2$

$p_2^{(R)} = 1$

$p_1^{(R)} = 0$

$p_1^{(L)} = 0$

$h_1^{(R)} = 0$

$h_2^{(R)} = 1$

$h_1^{(L)} = 0$

$h_2^{(L)} = 1$

Thermal form factor series for the transverse correlators

$$\left\langle \sigma_1^- \sigma_{m+1}^+(t) \right\rangle_T = \lim_{N\to+\infty} \sum_{k\ge 1} \mathrm{A}_k^{\sigma^-\sigma^+} \cdot \rho_k^m(0) \cdot \mathrm{e}^{-\mathrm{i}ht} \left(\frac{\rho_k(-\mathrm{i}t/\kappa N)}{\rho_k(\mathrm{i}t/\kappa N)} \right)^{\frac{N}{2}}$$

m&t dependent factors $\rightsquigarrow$ direct NLIE calculations

$$\rho_k^m(0) = -\mathrm{e}^{\mathrm{i}m\widehat{\mathcal{P}}(\widehat{\mathfrak{w}}_n)} \underset{N\to+\infty}{\longrightarrow} -\mathrm{e}^{\mathrm{i}m\mathcal{P}(\mathfrak{w}_n)} \quad \text{and} \quad \mathrm{e}^{-\mathrm{i}ht} \left(\frac{\rho_k(-\mathrm{i}t/\kappa N)}{\rho_k(\mathrm{i}t/\kappa N)} \right)^{\frac{N}{2}} = \mathrm{e}^{\mathrm{i}\widehat{\mathcal{E}}_t(\widehat{\mathfrak{w}}_n)} \underset{N\to+\infty}{\longrightarrow} \mathrm{e}^{\mathrm{i}t\mathcal{E}(\mathfrak{w}_n)}$$

with $\widehat{\mathfrak{w}}_n = (\widehat{\mathfrak{x}}_n ; \widehat{\mathfrak{y}}_{n-1})$, $\mathfrak{w}_n = (\mathfrak{x}_n ; \mathfrak{y}_{n-1})$ and for $\boldsymbol{w}_n = (\boldsymbol{x}_n, \boldsymbol{y}_{n-1})$

$$\begin{pmatrix} \mathcal{P}(\boldsymbol{w}_n) \\ \mathcal{E}(\boldsymbol{w}_n) \end{pmatrix} = \sum_{a=1}^{m} \begin{pmatrix} p_0(y_a) \\ \varepsilon_0(y_a) \end{pmatrix} - \sum_{a=1}^{n} \begin{pmatrix} p_0(x_a) \\ \varepsilon_0(x_a) \end{pmatrix} - \oint_{\mathscr{C}_u} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} \begin{pmatrix} p_0'(\lambda) \\ \varepsilon_0'(\lambda) \end{pmatrix} \mathscr{L}\mathrm{n}_{\mathscr{C}_u}\Big[1+\mathrm{e}^{-\frac{1}{T}u}\Big](\lambda \mid \boldsymbol{w}_n) - \begin{pmatrix} \mathcal{P}_{\mathrm{D}} \\ \mathcal{E}_{\mathrm{D}} \end{pmatrix}$$

Thermal form factor amplitude **'13-'14** **Dugave, Göhmann, K.**

$$\mathrm{A}_k^{\sigma^-\sigma^+} = \frac{(-1)^n \, \widehat{\mathscr{A}}^{-+}(\widehat{\mathfrak{w}}_n)}{\det\Big[\mathrm{D}_{\boldsymbol{w}_n}\Phi(\boldsymbol{w}_n)\Big]_{\boldsymbol{w}_n=\widehat{\mathfrak{w}}_n}}$$

$$\Phi(\boldsymbol{w}_n) = \Big(1+\mathrm{e}^{-\frac{1}{T}\widehat{u}(y_1|\boldsymbol{w}_n)}, \dots, 1+\mathrm{e}^{-\frac{1}{T}\widehat{u}(y_{n-1}|\boldsymbol{w}_n)} , \; 1+\mathrm{e}^{\frac{1}{T}\widehat{u}(x_1|\boldsymbol{w}_n)}, \dots, 1+\mathrm{e}^{\frac{1}{T}\widehat{u}(x_n|\boldsymbol{w}_n)}\Big)$$

Thermal form factor series in terms of multiple residues

⊛ Series of multidimensional residues '17 **Göhmann, Karbach, Klümper, K., Suzuki**

$$\left\langle \sigma_1^- \sigma_{m+1}^+(t) \right\rangle_T = \lim_{N \to +\infty} \sum_{n \geq 1} \frac{(-1)^{n+m}}{n!(n-1)!} \int_{\widehat{\mathscr{C}}_n} \frac{\mathrm{d}^{2n-1} w}{(2i\pi)^{2n-1}} \frac{\widehat{\mathscr{A}}^{-+}(\boldsymbol{w}) \cdot \mathrm{e}^{im\widehat{\Delta}(\boldsymbol{w})}}{\prod_{a=1}^{n-1} \left[1 + \mathrm{e}^{-\frac{1}{T}\widehat{u}(y_a | \boldsymbol{w})}\right] \cdot \prod_{a=1}^{n} \left[1 + \mathrm{e}^{\frac{1}{T}\widehat{u}(x_a | \boldsymbol{w})}\right]} .$$

⊛ Building blocks

- Thermal form factor amplitudes $\widehat{\mathscr{A}}^{-+}$;
- Complex phase $\widehat{\Delta}(\boldsymbol{w}) = \widehat{\mathcal{P}}(\boldsymbol{w}) + \frac{t}{m} \widehat{\mathcal{E}}(\boldsymbol{w})$;
- $\widehat{\mathscr{C}}_n$ integration manifold "surrounding" the BAE solutions;
- Denominator = residue locating factor.

Thermal form factor series at $N \to +\infty$

$$\langle \sigma_1^- \sigma_{m+1}^+(t) \rangle_T = \sum_{n \ge 1} \frac{(-1)^{n+m}}{n!(n-1)!} \int_{\mathscr{C}_n} \frac{\mathrm{d}^{2n-1} w}{(2\mathrm{i}\pi)^{2n-1}} \frac{\mathscr{A}^{-+}(\boldsymbol{w}) \cdot \mathrm{e}^{\mathrm{i} m \Delta(\boldsymbol{w})}}{\prod_{a=1}^{n-1} \Big[1 + \mathrm{e}^{-\frac{1}{T} u(y_a | \boldsymbol{w})} \Big] \cdot \prod_{a=1}^{n} \Big[1 + \mathrm{e}^{\frac{1}{T} u(x_a | \boldsymbol{w})} \Big]} .$$

- Rectifying map with $\boldsymbol{w} = (\boldsymbol{x}_n, \boldsymbol{y}_{n-1})$

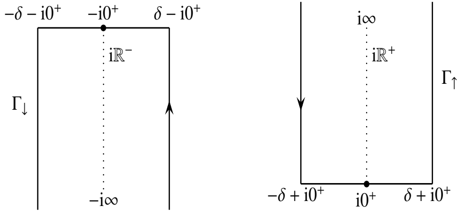
$$\Psi_n(\boldsymbol{w}) = \Big(u(y_1 \mid \boldsymbol{w}), \dots, u(y_{n-1} \mid \boldsymbol{w}), u(x_1 \mid \boldsymbol{w}), \dots, u(x_n \mid \boldsymbol{w}) \Big)$$

- Low-T control $\Psi_n = \Psi_n^{(0)} + \delta \Psi_n$ with

$$\Psi_n^{(0)}(\boldsymbol{w}) = \Big(\varepsilon_{\mathrm{c}}(y_1), \dots, \varepsilon_{\mathrm{c}}(y_{n-1}), \varepsilon_{\mathrm{c}}(x_1), \dots, \varepsilon_{\mathrm{c}}(x_n) \Big) \qquad \text{and} \qquad \delta \Psi_n = \mathrm{O}(T)$$

- Integration curve $\mathscr{C}_n = \Psi_n^{-1}\Big(\Gamma_{\mathrm{tot}}^{2n-1} \Big)$ with $\Gamma_{\mathrm{tot}} = \Gamma_\uparrow \cup \Gamma_\downarrow$

The rectified integration curve



Thermal form factor series at $N \to +\infty$

$$\left\langle \sigma_1^- \sigma_{m+1}^+(t) \right\rangle_T = \sum_{n \geq 1} \frac{(-1)^{n+m}}{n!(n-1)!} \int_{\mathscr{C}_n} \frac{\mathrm{d}^{2n-1} w}{(2i\pi)^{2n-1}} \frac{\mathscr{A}^{-+}(\boldsymbol{w}) \cdot \mathrm{e}^{\,im\Delta(\boldsymbol{w})}}{\prod_{a=1}^{n-1} \left[1 + \mathrm{e}^{-\frac{1}{T} u(y_a | \boldsymbol{w})}\right] \cdot \prod_{a=1}^{n} \left[1 + \mathrm{e}^{\frac{1}{T} u(x_a | \boldsymbol{w})}\right]} .$$

⊛ Amplitude function $\mathscr{A}^{-+}(\boldsymbol{w})$:

- symmetric in respect to $\boldsymbol{x}_n$ and $\boldsymbol{y}_{n-1}$;
- vanishes on "x" or "y" diagonal: $\mathscr{A}^{-+}(\boldsymbol{w})_{|x_a = x_b} = 0 = \mathscr{A}^{-+}(\boldsymbol{w})_{|y_a = y_b}$.

⊛ Complex valued phase $\Delta(\boldsymbol{w}) = \mathcal{P}(\boldsymbol{w}) + \frac{t}{m}\mathcal{E}(\boldsymbol{w}) = \Delta_0(\boldsymbol{w}) + \mathrm{O}(T)$

$$\Delta_0(\boldsymbol{w}) = \sum_{a=1}^{n-1} \Big\{ p(y_a) + \frac{t}{m}\varepsilon(y_a) \Big\} - \sum_{a=1}^{n} \Big\{ p(x_a) + \frac{t}{m}\varepsilon(x_a) \Big\}$$

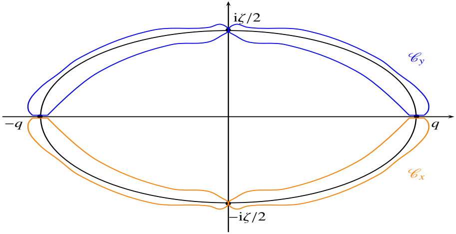
⇝ Low-T expansion in terms of dressed energy and momentum.

Towards the large-m, $|t| < \mathfrak{c}m$, analysis

- Regime $|t| < \mathfrak{c}m$, T-low
  - $\Delta(\boldsymbol{w})$ away of $\pm \mathrm{i}\zeta/2$ controlled by $p(\lambda)$

    $$\Im[p(\lambda)] > 0 \quad \text{if} \quad 0 < \Im(\lambda) < \pi/2 \qquad \textit{and} \qquad \Im[p(\lambda)] < 0 \quad \text{if} \quad -\pi/2 < \Im(\lambda) < 0$$

  - Saddle point $p'(\lambda) + \frac{t}{m}\varepsilon'(\lambda) = 0$ away from real axis
  - "x" and "y" type curves in $T = 0$ approximation $\mathscr{C}_{n|T=0} = \mathscr{C}_y^{n-1} \times \mathscr{C}_x^n$



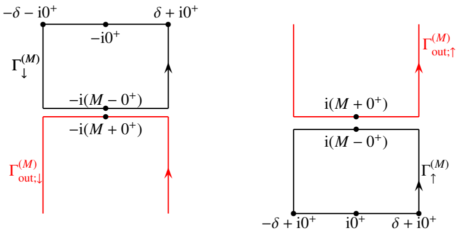
⊛ Deform y contour to $\Im(\lambda) > 0$ and x contour to $\Im(\lambda) < 0$

⇝ Denominators generate pole contributions

Towards the large-m, $|t| < \mathfrak{c}m$, analysis

- Decompose original contour into "close" and "far" poles contributions

$$\mathscr{C}_n = \Psi_n^{-1}\Big(\big\{\Gamma_{\downarrow}^{(M)} \cup \Gamma_{\mathrm{out};\downarrow}^{(M)} \cup \Gamma_{\uparrow}^{(M)} \cup \Gamma_{\mathrm{out};\uparrow}^{(M)}\big\}^{2n-1}\Big)$$



- Compute explicitly all "close" poles contributions

Towards the large-m, $|t| < \mathfrak{c}m$, analysis

$$\Big\langle \sigma_1^- \sigma_{m+1}^+(t) \Big\rangle_T = \sum_{n \ge 1} \sum_{\substack{\sum n_p^{(\upsilon)} \\ = n-1}} \sum_{\substack{\sum n_h^{(\upsilon)} \\ = n}} \frac{1}{n_p^{(M)}!\, n_h^{(M)}!} \prod_{\sigma=\pm} \Bigg\{ (-T)^{n_h^{(\sigma)}} \sum_{0 \le h_1^{(\sigma)} < \dots < h_{n_h^{(\sigma)}}^{(\sigma)}}^{M-1} T^{n_p^{(\sigma)}} \sum_{0 \le p_1^{(\sigma)} < \dots < p_{n_p^{(\sigma)}}^{(\sigma)}}^{M-1} \Bigg\}$$

$$\times \int_{\mathscr{C}_{n_p^{(M)}, n_h^{(M)}}} \frac{\mathrm{d}^{n_p^{(M)}} y^{(M)}}{(2\mathrm{i}\pi)^{n_p^{(M)}}} \cdot \frac{\mathrm{d}^{n_h^{(M)}} x^{(M)}}{(2\mathrm{i}\pi)^{n_h^{(M)}}} \frac{\det\big[\mathrm{D}\Psi_{n;\mathrm{out}}(\varpi_{\boldsymbol{p},\boldsymbol{h}})\big]}{\det\big[\mathrm{D}\Psi_{n}(\varpi_{\boldsymbol{p},\boldsymbol{h}})\big]} \cdot \frac{(-1)^{m+n} \mathscr{A}^{-+}\big(\varpi_{\boldsymbol{p},\boldsymbol{h}}\big) \cdot \mathrm{e}^{\mathrm{i}m\Delta(\varpi_{\boldsymbol{p},\boldsymbol{h}})}}{\prod\limits_{a=1}^{n_p^{(M)}} \Big[1 + \mathrm{e}^{-\frac{1}{T} u(y_a^{(M)} | \varpi_{\boldsymbol{p},\boldsymbol{h}})}\Big] \cdot \prod\limits_{a=1}^{n_h^{(M)}} \Big[1 + \mathrm{e}^{\frac{1}{T} u(x_a^{(M)} | \varpi_{\boldsymbol{p},\boldsymbol{h}})}\Big]}$$

⊛ Integration involves vectors

$$\varpi_{\boldsymbol{p},\boldsymbol{h}} = \Big(\mathrm{x}_{n_h^{(-)}}^{(-)}, \mathrm{x}_{n_h^{(+)}}^{(+)}, \boldsymbol{x}_{n_h^{(M)}}^{(M)}; \mathrm{y}_{n_p^{(-)}}^{(-)}, \mathrm{y}_{n_p^{(+)}}^{(+)}, \boldsymbol{y}_{n_p^{(M)}}^{(M)} \Big)^{\mathrm{t}}$$

built out of solutions to partial quantisation conditions

$$u\Big(\mathrm{x}_a^{(\sigma)} \mid \varpi_{\boldsymbol{p},\boldsymbol{h}}\Big) = -\sigma 2\mathrm{i}\pi T\Big(h_a^{(\sigma)} + \tfrac{1}{2}\Big) \quad , \quad a = 1, \dots, n_h^{(\sigma)}$$

$$u\Big(\mathrm{y}_a^{(\sigma)} \mid \varpi_{\boldsymbol{p},\boldsymbol{h}}\Big) = \sigma 2\mathrm{i}\pi T\Big(p_a^{(\sigma)} + \tfrac{1}{2}\Big) \quad , \quad a = 1, \dots, n_p^{(\sigma)}$$

Leading contribution in the low-T regime

- Integrations along $\mathscr{C}_{n_p^{(M)}, n_h^{(M)}}$ produces sub-dominant contributions
- Pure pole contributions $\hookrightarrow$ $\boldsymbol{\varpi}_{\boldsymbol{p},\boldsymbol{h}} = \Big(\mathrm{x}_{n_h^{(-)}}^{(-)}, \mathrm{x}_{n_h^{(+)}}^{(+)} ; \mathrm{y}_{n_p^{(-)}}^{(-)}, \mathrm{y}_{n_p^{(+)}}^{(+)} \Big)^{\mathrm{t}}$
- Low-T complex valued phase expansion

$$\Delta(\boldsymbol{\varpi}_{\boldsymbol{p},\boldsymbol{h}}) = (\ell^{(-)} + \ell^{(+)} + 1)p(q) + T\Big[\Delta_0^{(1)}(\boldsymbol{\varpi}_{\boldsymbol{p},\boldsymbol{h}}) + \Delta_0^{(2)}(\boldsymbol{\varpi}_{\boldsymbol{p},\boldsymbol{h}})\Big] + \mathrm{O}(T^2)$$

in terms of excitation integers

$$\Delta_0^{(1)}(\boldsymbol{\varpi}_{\boldsymbol{p},\boldsymbol{h}}) = \frac{2\mathrm{i}\pi}{\mathrm{v}_F}\bigg\{ \Big(Z(q)(n_h^{(-)} - n_p^{(-)}) - \frac{\mathrm{v}_F t}{2mZ(q)} \Big)^2 + \frac{1}{4Z^2(q)}\Big(1 - \big(\frac{\mathrm{v}_F t}{m}\big)^2\Big) + \sum_{\sigma=\pm} n_p^{(\sigma)} n_h^{(\sigma)} \Big(1 + \sigma \frac{\mathrm{v}_F t}{m}\Big) \bigg\}$$

$$\Delta_0^{(2)}(\boldsymbol{\varpi}_{\boldsymbol{p},\boldsymbol{h}}) = \frac{2\mathrm{i}\pi}{\mathrm{v}_F} \sum_{\sigma=\pm} \Big(1 + \sigma \frac{\mathrm{v}_F t}{m}\Big) \Big[\sum_{a=1}^{n_p^{(\sigma)}} \big(p_a^{(\sigma)} - a + 1\big) + \sum_{a=1}^{n_h^{(\sigma)}} \big(h_a^{(\sigma)} - a + 1\big) \Big]$$

- Minimal value of $\Im\Big[\Delta_0^{(1)} + \Delta_0^{(2)}\Big]$ attained at

$$n_p^{(-)} = n_p^{(+)} = n_h^{(-)} = 0 \,, n_h^{(+)} = 1 \quad \textit{and} \quad h_1^{(+)} = 0$$

with value

$$(\Delta_0^{(1)} + \Delta_0^{(2)})(\boldsymbol{\varpi}_{\boldsymbol{p}_{\min},\boldsymbol{h}_{\min}}) = \frac{\mathrm{i}\pi}{2\mathrm{v}_F Z^2(q)}$$

Leading $(m,t) \to \infty$ space-like asymptotics with $|t| < \mathfrak{c}m$

- Leading $(m,t) \to \infty$ behaviour for $|t| < \mathfrak{c}m$

$$\left\langle \sigma_1^- \sigma_{m+1}^+(t) \right\rangle_T = (-1)^m T \frac{\mathscr{A}^{-+}(\boldsymbol{w}_{\mathrm{dom}})}{u'(x \mid \boldsymbol{w}_{\mathrm{dom}})} \mathrm{e}^{\mathrm{i}m\Delta(\boldsymbol{w}_{\mathrm{dom}})} \cdot \left(1 + \mathrm{O}(m^{-\infty})\right)$$

- Dominant asymptotics parameterised by 1-dimensional vector

$$\boldsymbol{w}_{\mathrm{dom}} = \left(\boldsymbol{y}_{\mathrm{dom}}, \boldsymbol{x}_{\mathrm{dom}}\right)^{\mathrm{t}} \quad \text{with} \quad \boldsymbol{y}_{\mathrm{dom}} = \emptyset\,, \ \boldsymbol{x}_{\mathrm{dom}} = (x)$$

- $\left(x, u(\lambda \mid \boldsymbol{w}_{\mathrm{dom}})\right)$ solve the non-linear system

$$u(x \mid \boldsymbol{w}_{\mathrm{dom}}) = -\mathrm{i}\pi T\,, \quad \Re(x) > 0$$

$$u(\lambda \mid \boldsymbol{w}_{\mathrm{dom}}) = \varepsilon_0(\lambda) + \mathrm{i}T\theta(\lambda - x) - T\int_{\mathscr{C}_u} \frac{\mathrm{d}\mu}{2\pi}\theta'(\lambda-\mu)\mathscr{L}n\left[1 + \mathrm{e}^{-\frac{1}{T}u}\right](\mu \mid \boldsymbol{w}_{\mathrm{dom}})$$

- Correlation length given by effective momentum and energy

$$\Delta(\boldsymbol{w}_{\mathrm{dom}}) = \mathcal{P}(\boldsymbol{w}_{\mathrm{dom}}) + \frac{t}{m}\mathcal{E}(\boldsymbol{w}_{\mathrm{dom}})$$

- Amplitude given by thermal form factors

Conclusion and perspectives

Review of the results

- ✔ Rigorous understanding of the BAE related NLIE in low-T large-N regime;
- ✔ Leading term of the $|t| < \mathfrak{c}m$ asymptotics.

Further developments

- ⊛ Understand the whole $(m, t) \to \infty$ regime at finite T.

Happy Birthday Fedor