Supersymmetry and trace formulas: Selberg trace formula

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(Based on the joint work with Changha Choi, arXiv:2112.07942 & arXiv:2306.13636)

I. Introduction

- Supersymmetry, a global symmetry between bosons and fermions, provides invaluable insights to the non-perturbative aspects of general strongly coupled quantum field theories, and is deeply related to various areas of mathematics.

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- The Witten index

$$I = \mathrm{Str}\, e^{-\beta \hat{H}} = \mathrm{Tr}(-1)^{F} e^{-\beta \hat{H}}$$

gives precise non-perturbative information about the ground states of a supersymmetric quantum Hamiltonian $\hat{H}$.

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  gives precise non-perturbative information about the ground states of a supersymmetric quantum Hamiltonian $\hat{H}$.
- I is related to the index of the Dirac operator and is computed by supersymmetric localization, the infinite-dimensional version of the Duistermaat-Heckman formula.

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- The Witten index is given by the path integral

$$I = \mathrm{Tr}(-1)^F e^{-\beta \hat{H}} = \int e^{-S_E[x,\psi]} \mathscr{D}x \mathscr{D}\psi,$$

where

$$S_E[x, \psi] = \int_0^\beta \mathcal{L}_E(x, \dot{x}; \psi, \dot{\psi}) dt$$

is the Euclidean action, and $\mathscr{D}x\mathscr{D}\psi$ is path integration 'measure' for the bosonic and fermionic degrees of freedom.

- The integration goes over periodic boundary conditions and

$$\delta S_E = 0 \quad \text{and} \quad \delta(\mathscr{D}x\mathscr{D}\psi) = 0.$$

Here δ is the Wick rotated classical supersymmetry transformation generated by a supercharge Q,

$$\delta x^\mu = \{Q, x^\mu\} = \psi^\mu, \quad \delta\psi^\mu = \{Q, \psi^\mu\} = -\dot{x}^\mu.$$

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- Let $V[x, \psi]$ be an invariant deformation, a functional of classical fields satisfying

$$\delta^2 V = 0.$$

The key fact: for all real λ we have

$$\boxed{\int e^{-S_E}\mathscr{D}x\mathscr{D}\psi = \int e^{-S_E - \lambda\delta V}\mathscr{D}x\mathscr{D}\psi}$$

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- In case $S_E = \delta V$ the path integral in the limit $\lambda \to \infty$ localizes on the zero locus of S_E. The latter is the set of constant loops, arising from the standard kinetic term in the action.

2. Example

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- The Witten index ($\mathcal{L}(M)$ is a free loop space of M)

$$I = \operatorname{Str} e^{-\beta\hat{H}} = \int_{\Pi T\mathcal{L}(M)} e^{-S_E}\mathscr{D}x\mathscr{D}\psi$$

  localizes on constant loops (Witten 1982, Atiyah 1985); explicit computation (L. Alvarez-Gaumé, 1983) gives Atiyah-Singer formula for the index of Dirac operator.

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    2. In the Hilbert space $\mathscr{H}$ the Majorana fermions $\hat{\chi}_1, \ldots, \hat{\chi}_n$ satisfy

$$\hat{\chi}_1 \cdots \hat{\chi}_n = 2^{-\frac{n}{2}} (-1)^F,$$

so

$$\operatorname{Str} \hat{\chi}_1 \cdots \hat{\chi}_n e^{-\beta \hat{H}} = 2^{-\frac{n}{2}} \operatorname{Tr} e^{-\beta \hat{H}} = \int \chi_1 \cdots \chi_n e^{-S_E} \mathscr{D}x \mathscr{D}\psi.$$

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- However, the path integral nontrivially depends on β and since $\delta(\chi_1 \cdots \chi_n e^{-S_E}) \neq 0$, standard localization does not apply.

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- Note that condition **(A)** is rather natural, condition **(B)** is standard, while condition **(C)**, the absence of fermion zero modes in V and δV, is a completely new requirement. It is rather constraining and forces V to explicitly depend on the first time derivatives of fermion degrees of freedom.

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- **Let S_E be the Euclidean action of the supersymmetric theory with fermion zero modes $\chi_1, \ldots, \chi_n$ satisfying conditions 1-2 and (A). Then for all λ we have**

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where V is a deformation satisfying conditions (B)–(C).

- If bosonic and fermionic degrees of freedom decouple

$$\mathscr{H} = \mathscr{H}_B \otimes \mathscr{H}_F \quad \text{and} \quad \hat{H} = \hat{H}_B \otimes I_F + I_B \otimes \hat{H}_F,$$

then

$$\operatorname{Str} \hat{\chi}_1 \cdots \hat{\chi}_n e^{-\beta \hat{H}} = 2^{-n/2} \operatorname{Tr}_{\mathscr{H}_F} e^{-\beta \hat{H}_F} \cdot \operatorname{Tr}_{\mathscr{H}_B} e^{-\beta \hat{H}}.$$

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- If $\hat{H}_F = 0$, we have

$$\boxed{\mathrm{Str}\, \hat{\chi}_1 \cdots \hat{\chi}_n e^{-\beta\hat{H}} = \mathrm{Tr}_{\mathscr{H}_B}\, e^{-\beta\hat{H}_B}}$$

Thus we obtain a pure bosonic trace formula by localizing the supersymmetric path integral in the limit $\lambda \to \infty$ to the zero locus of V.

III. Examples

1. Poisson summation formula: localization on $\mathrm{U}(1)$

- Free supersymmetric particle of mass $m = 1$ on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ with the Lagrangian, the real supercharge

$$\mathcal{L} = \frac{1}{2}(\dot{x}^2 + i\psi\dot{\psi}), \qquad Q = i\dot{x}\psi$$

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- New localization principle: the path integral

$$\int_{\Pi T \mathcal{L}(S^1)} \chi e^{-S_E + \lambda \delta V} \mathscr{D}x \mathscr{D}\psi,$$

  where

$$V = \frac{1}{2} \int_0^\beta \ddot{x} \dot{\psi} \, dt, \quad \delta V = -\frac{1}{2} \int_0^\beta (\ddot{x}^2 + \dot{\psi} \ddot{\psi}) dt,$$

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  does not depend on λ!

- In the limit $\lambda \to \infty$ the path integral localizes on the classical trajectories $\ddot{x} = 0$, and one can compute $Z(\beta)$ exactly.

- Specifically, we obtain

$$\boxed{\sum_{n=-\infty}^{\infty} e^{-n^2\beta/2}} = 2\pi \lim_{s\to\infty} \int_{\Pi T\Omega S^1} e^{-S_E - s\delta V} \mathscr{D}' x \mathscr{D}' \psi$$

$$= 2\pi \cdot (2\pi)^{\zeta(0)} \int_{\Pi T\Omega S^1} e^{-S_E} \delta(\ddot{x})\delta(\psi)\mathrm{Pf}(\partial_t^3)\mathscr{D}' x \mathscr{D}' \psi$$

$$= 2\pi \cdot (2\pi)^{\zeta(0)} \int_{\Omega S^1} e^{-S_E[x,0]} \sum_{x_{cl}} \frac{\delta(x - x_{cl})}{\det(\partial_t^2)} \mathrm{Pf}(\partial_t^3)\mathscr{D}' x$$

$$= 2\pi \cdot (2\pi)^{\zeta(0)} \sum_{x_{cl}} e^{-\frac{1}{2}\int_0^\beta \dot{x}_{cl}^2 dt} \frac{\mathrm{Pf}(\partial_t^3)}{\det(\partial_t^2)}$$

$$= \boxed{\sqrt{\frac{2\pi}{\beta}} \sum_{n=-\infty}^{\infty} e^{-2\pi^2 n^2/\beta}}$$

which is Jacobi inversion formula.

2. Eskin summation formula: localization on G

- This summation formula was first obtained by L.D. Eskin (Л.Д. Эскин "Уравнение теплопроводности на группах Ли", Сб. памяти Н.Г. Чеботарева, Изд. КГУ, Казань, 1964; см. также Л.Д. Эскин "Уравнение теплопроводности в теории компактных групп", УМН, **19**:2(116) (1964), 200–202), and rediscovered later by I. Frenkel and J.-M. Bismut.

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- $0+1$ supersymmetric sigma model — supersymmetric particle on compact simple Lie group G with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\langle \dot{x}, \dot{x}\rangle + \frac{i}{2}\langle \boldsymbol{\psi}, \nabla^-_{\dot{x}}\boldsymbol{\psi}\rangle, \quad \boldsymbol{\psi} \in \Pi T_{x(t)}G,$$

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  where ∇^- is flat left-invariant connection on G (with torsion).
- In Cartan moving frame formalism $J = g^{-1}\dot{g} \in \mathfrak{g}$ and $\psi = L_{g^{-1}}\boldsymbol{\psi} \in \Pi\mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G and

$$\mathcal{L} = \frac{1}{2}\langle J, J \rangle + \frac{i}{2}\langle \psi, \dot{\psi} \rangle.$$

- Real supercharge

$$Q = \langle \psi, J \rangle + \frac{i}{6} \langle \psi, [\psi, \psi] \rangle$$

and classical Hamiltonian

$$H = \frac{1}{2i} \{Q, Q\} = \frac{1}{2} g^{ab} l_a l_b$$

with the Dirac brackets on the reduced phase space

$$\{p_\mu, x^\nu\} = \delta_\mu^\nu \quad \text{and} \quad \{\psi^a, \psi^b\} = i g^{ab}.$$

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- Quantization $\mathscr{H} = L^2(G) \otimes \mathscr{H}_F$,

$[\hat{\psi}^a, \hat{\psi}^b] = g^{ab},\ [\hat{l}_a, \hat{l}_a] = -i f_{ab}^c \hat{l}_c$ and $\hat{Q} = \hat{\psi}^a \hat{l}_a + \dfrac{i}{6} f_{abc} \hat{\psi}^a \hat{\psi}^b \hat{\psi}^c$.

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$[\hat{\psi}^a, \hat{\psi}^b] = g^{ab}, \ [\hat{l}_a, \hat{l}_a] = -i f_{ab}^c \hat{l}_c$ and $\hat{Q} = \hat{\psi}^a \hat{l}_a + \frac{i}{6} f_{abc} \hat{\psi}^a \hat{\psi}^b \hat{\psi}^c$.

- Hamiltonian operator $\hat{H} = \hat{Q}^2$ is given by

$$\hat{H} = \frac{1}{2} g^{ab} \hat{l}_a \hat{l}_b + \frac{1}{48} f_{abc} f^{abc} \hat{I} = \frac{1}{2} \Delta + \frac{R}{12} \hat{I},$$

where Δ is the Laplace operator on $L^2(G)$ and the second term is the 'notorious' DeWitt term.

- Fermion zero modes

$$\chi^a = \frac{1}{\beta}\int_0^\beta \psi^a \, dt,$$

so

$$\mathsf{Str}\,\hat{\chi}^1 \ldots \hat{\chi}^n e^{-\beta\hat{H}} = e^{-\frac{1}{12}\beta R}\,\mathrm{Tr}\,e^{-\frac{1}{2}\beta\Delta}.$$

and

$$\mathsf{Str}\,\hat{\chi}^1 \ldots \hat{\chi}^n e^{-\beta\hat{H}+i\langle h,\hat{r}\rangle} = V_G e^{-\frac{1}{12}\beta R} K_\beta(e^h),$$

where K_β is the heat kernel, $\hat{r} = \hat{r}^a T_a$ and $h \in \mathfrak{t}$.

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$$\chi^a = \frac{1}{\beta}\int_0^\beta \psi^a\, dt,$$

so

$$\mathsf{Str}\, \hat{\chi}^1 \dots \hat{\chi}^n e^{-\beta \hat{H}} = e^{-\frac{1}{12}\beta R}\, \mathrm{Tr}\, e^{-\frac{1}{2}\beta\Delta}.$$

and

$$\mathsf{Str}\, \hat{\chi}^1 \dots \hat{\chi}^n e^{-\beta \hat{H} + i\langle h, \hat{r}\rangle} = V_G e^{-\frac{1}{12}\beta R} K_\beta(e^h),$$

where K_β is the heat kernel, $\hat{r} = \hat{r}^a T_a$ and $h \in \mathfrak{t}$.

- Path integral representation

$$\mathsf{Str}\, \hat{\chi}^1 \dots \hat{\chi}^n e^{-\beta \hat{H} + i\langle h, \hat{r}\rangle} = \int\limits_{\Pi TLG} \chi^1 \dots \chi^n e^{-S_E^h}\, \mathscr{D}g \mathscr{D}\psi,$$

where

$$S_E^h = \frac{1}{2}\int_0^\beta (\langle J, J\rangle + \langle \psi, \dot{\psi}\rangle) dt + \frac{1}{\beta}\int_0^\beta \langle \mathrm{Ad}_{g^{-1}} h, J\rangle dt.$$

- The supersymmetric deformation is

$$V = -\frac{1}{2}\int_0^\beta \langle \dot{J}^h, \dot{\psi} \rangle dt$$

where

$$J^h = J + \frac{1}{\beta}\mathsf{Ad}_{g^{-1}} h$$

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- When $h \in \mathfrak{t}$ is regular, on ΩG solutions are isolated geodesics and one computes the supertrace

•

$$\mathsf{Str}\,\hat{\chi}^1 \ldots \hat{\chi}^n e^{-\beta\hat{H}+i\langle h,\hat{r}\rangle}$$

$$= \frac{V_G}{(2\pi\beta)^{n/2}} \sum_{\gamma\in\Gamma} \prod_{\alpha\in R_+} \frac{\frac{1}{2}\langle\alpha, h+\gamma\rangle}{\sin\frac{1}{2}\langle\alpha, h+\gamma\rangle} e^{-\frac{1}{2\beta}\langle h+\gamma, h+\gamma\rangle}$$

and we obtain the Eskin formula for the heat kernel

$$K_\beta(e^h) = \frac{e^{\frac{1}{2}\beta\langle\rho,\rho\rangle}}{(2\pi\beta)^{n/2}} \sum_{\gamma\in\Gamma} \prod_{\alpha\in R_+} \frac{\frac{1}{2}\langle\alpha, h+\gamma\rangle}{\sin\frac{1}{2}\langle\alpha, h+\gamma\rangle} e^{-\frac{1}{2\beta}\langle h+\gamma, h+\gamma\rangle},$$

where $\Gamma = \{\gamma \in \mathfrak{t} : e^\gamma = 1\}$ is the characteristic lattice, which is related to the maximal torus by $T = \mathfrak{t}/\Gamma$.

- $$\mathsf{Str}\,\hat{\chi}^1 \ldots \hat{\chi}^n e^{-\beta \hat{H} + i\langle h, \hat{r}\rangle}$$
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- Comparing with the spectral representation

$$K_\beta(e^h) = \frac{1}{V_G} \sum_{\pi \in \mathrm{Irrep}\, G} d_\pi\, \chi_\pi(h) e^{-\frac{1}{2}\beta C_2(\pi)},$$

  we obtain Eskin summation formula.

3. Selberg trace formula: localization on $\Gamma\backslash G/K$

Example: $G = \mathrm{SL}(2,\mathbb{R})$, $K = \mathrm{SO}(2)$ and Γ is a discrete subgroup of G containing $-I$, so $X = \Gamma\backslash G/K$ is compact hyperbolic Riemann surface (with orbifold points).

- Supersymmetric sigma model on $\Gamma\backslash G$

$$\mathcal{L} = \frac{1}{2}\langle J, J\rangle + \frac{i}{2}\langle \psi, \dot{\psi}\rangle$$

in Lorentzian time $0 \leq t \leq T$, using Cartan frame formalism $J = g^{-1}\dot{g}$ and $\psi = L_g^{-1}\boldsymbol{\psi}$;

$$\delta g = ig\psi \quad \text{and} \quad \delta\psi = -J - i\psi\psi.$$

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- The Hilbert space is

$$\mathscr{H}_{\Gamma\backslash G} = L^2(\Gamma\backslash G, dg) \otimes \mathscr{H}_{F,\mathfrak{g}},$$

  but we need the Hilbert space $L^2(X, d\mu_{\mathrm{hyp}})$. It can be obtained by gauging the right K-symmetry $g \mapsto gk$ and $\psi \mapsto \mathrm{Ad}_{k^{-1}}\psi$, $k \in K$, by using a K-connection A in the principal bundle $K \to S^1_T = \mathbb{R}/T\mathbb{Z}$.

- Gauged sigma model on $\Gamma\backslash G$

$$\mathcal{L}_0 = \frac{1}{2}\langle J_A, J_A\rangle + \frac{i}{2}\langle\psi, \partial_t^A\psi\rangle,$$

where $J_A = J - A$ and $\partial_t^A = \partial_t + \mathrm{ad}_A$.

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- Since the Lagrangian $\mathcal{L}_0$ has no kinetic term for A, we have a classical Gauss law

$$C_0 : \; J_A^3 + 2i\psi^1\psi^2 = 0,$$

which is realized quantum mechanically as the constraint on the Hilbert space $\mathscr{H}_{\Gamma\backslash G}$.

- The main representation

$$Z(iT) = \mathrm{Tr}_{L^2(X)}[e^{-iT\Delta/2}]$$

$$= \frac{e^{-\frac{i\langle\rho,\rho\rangle T}{2}}}{\mathrm{vol}(\mathcal{G})} \int \frac{1}{W_{-1}(A) - W_1(A)} \psi_0^3 e^{i\int_0^T \mathcal{L}_0 dt} \mathscr{D}g \mathscr{D}\psi \mathscr{D}A,$$

where domain of integration is

$$L(\Gamma\backslash G) \times \Pi L\mathfrak{g} \times \mathcal{A}.$$

Here $\mathcal{G}$ is the gauge group, t_1, t_2, t_3 are generators of $\mathfrak{g}$, t_3 — generator of $\mathfrak{k}$, $A = A^3 t_3$,

$$\psi_0^3 = \frac{1}{T}\int_0^T \psi^3(t)dt$$

is fermion zero mode and

$$W_{\pm 1}(A) = e^{\pm i\int_0^T A^3(t)dt}$$

are Wilson lines.

- Connected components of the free loop space $L(\Gamma\backslash G)$ are parametrized by the conjugacy classes $[\gamma]$ of the elements $\gamma \in \Gamma$, and we obtain the 'pre-trace' formula

$$Z(iT) = \sum_{[\gamma]} Z_{[\gamma]}(iT),$$

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- The new supersymmetric localization principle allows to compute explicitly each orbital integral in the pre-trace formula in the limit $\lambda \to \infty$.
- We have $Z_{[\gamma]}(iT) = Z_{[-\gamma]}(iT)$; computing $Z_{[\gamma]}(iT)$ for the identity, hyperbolic and elliptic elements, and performing the Wick rotation $T \mapsto -i\beta$, we obtain the Selberg trace formula (with exact match of all coefficients)!

Рис.: Тянджинь, Нанкай, 1989

Рис.: Вена, 2004

Рис.: Женева, 2009

Happy Birthday, Fedya!