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Scientific Achievements of Fedor Smirnov

Michio Jimbo

Integrable Systems and Field Theory

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0. Introduction

Over the last 40 years, F.Smirnov has made a number of original and important contributions in integrable systems and filed theory.

The aim of this talk is to quickly go through his major achievements, trying with the audience to follow the path how his thoughts have shaped and developed.

Plan

1. **Form factor bootstrap** $\sim$ **1990**
2. **qKZ, classical limit, and BBS fermions** $\sim$ **2000**
3. **Fermionic structure** **2000** $\sim$
   (a) **Lattice models**
      - **Creation/annihilation operators**
      - **Main theorem**
      - **Inverse problem**

   (b) **Field theory**
      - **Fermions in CFT and reflection relations**
      - **One point functions in QFT**

1. Form Factor Bootstrap

Early developments

- **Quantization of solitons** — **Faddeev-Korepin 1978**
- **Factorized S matrices** — **Zamolodchikov-Zamolodchikov 1979**
- **QISM** — **Fadeev, Skyanin, Takhtajan 1979**

Smirnov formulated the quantum Gelfand-Levitan equation and obtained soliton form factors in the sine-Gordon model [6]–[9].

$$\mathcal{A}^{\mathrm{sG}} = \int \Big\{ \frac{1}{4\pi} \partial_z \varphi(z, \bar{z}) \partial_{\bar{z}} \varphi(z, \bar{z}) - \frac{2\mu^2}{\sin \pi\beta^2} \cos(\beta\varphi(z, \bar{z})) \Big\} d^2 z\,.$$

we will switch from β *to* $p = \beta^2/(1 - \beta^2)$

This led him to axiomatization for form factors [12]–[14].

A form factor is a matrix element of a local field $\mathcal{O}_\alpha$

$$f_{\mathcal{O}_\alpha}(\beta_1, \ldots, \beta_{2n}) = {}_{in}\langle \beta_1, \ldots, \beta_{2n} | \mathcal{O}_\alpha(0) | 0 \rangle$$

between vacuum and an in-state with rapidities $\beta_1, \ldots, \beta_{2n}$ (suffix α indicates that we consider $e^{i\tilde{\alpha}\varphi}$, $\tilde{\alpha} = \alpha/(2\sqrt{p(p+1)})$, and its descendants).

Given the 2-particle S matrix

$$S(\beta) = S_0(\beta) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{\sinh \frac{1}{p}\beta}{\sinh \frac{1}{p}(\beta - \pi i)} & \frac{\sinh \frac{\pi i}{p}}{\sinh \frac{1}{p}(\beta - \pi i)} & 0 \\ 0 & \frac{\sinh \frac{\pi i}{p}}{\sinh \frac{1}{p}(\beta - \pi i)} & -\frac{\sinh \frac{1}{p}\beta}{\sinh \frac{1}{p}(\beta - \pi i)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

as an input, form factors

$$f_{\mathcal{O}_\alpha}(\beta_1, \ldots, \beta_{2n}) \in \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2\,, \quad n = 0, 1, 2, \ldots,$$

satisfy the following axioms which ensure the locality of the field $\mathcal{O}_\alpha$.

1)Analyticity

$f_{\mathcal{O}_\alpha}(\beta_1,\ldots,\beta_{2n})$ is meromorphic. In $0 \le \mathrm{Im}(\beta_{2n}) < 2\pi$, only simple poles at $\beta_{2n} = \beta_j + \pi i$, $j \neq 2n$ (and possibly also those for breathers).

2)Symmetry

$$
\begin{aligned}
&S(\beta_j - \beta_{j+1}) f_{\mathcal{O}_\alpha}(\beta_1,\ldots,\beta_j,\beta_{j+1},\ldots,\beta_{2n}) \\
&\qquad\qquad = f_{\mathcal{O}_\alpha}(\beta_1,\ldots,\beta_{j+1},\beta_j,\ldots,\beta_{2n})\,.
\end{aligned}
$$

3)Riemann-Hilbert problem

$$
f_{\mathcal{O}_\alpha}(\beta_1,\ldots,\beta_{2n-1},\beta_{2n}+2\pi i) = e^{-\frac{\alpha}{p}\pi i\sigma^3_{2n}} f_{\mathcal{O}_\alpha}(\beta_{2n},\beta_1,\ldots,\beta_{2n-1})\,.
$$

4)Residue

$$
\begin{aligned}
&2\pi i \operatorname*{res}_{\beta_{2n}=\beta_{2n-1}+\pi i} f_{\mathcal{O}_\alpha}(\beta_1,\ldots,\beta_{2n-2},\beta_{2n-1},\beta_{2n}) = \\
&\Bigl(1 - e^{-\frac{\alpha}{p}\pi i\sigma^3_{2n}} S(\beta_{2n-1}-\beta_{2n-2})\cdots S(\beta_{2n-1}-\beta_1)\Bigr) \\
&\times f_{\mathcal{O}_\alpha}(\beta_1,\ldots,\beta_{2n-2}) \otimes s_{2n-1,2n}\,.
\end{aligned}
$$

These axioms ensure that the corresponding field is local [12].

His book [23]

Form Factors in Completely Integrable Models of Quantum Field Theory

Advanced Series in Mathematical Physics 14,

World Scientific, Singapore, 1992.

gives an exposition of these results in detail.

2. qKZ, Classical limit, and BBS fermions

Axioms (2) and (3) imply that form factors satisfy the qKZ equation *at level* **0; this was the first instance qKZ appeared in the literature.**

I.Frenkel and Reshetikhin, 1992

A very special feature of the level being 0 is that solutions are written as determinants of *one-dimensional* **integrals.**

The essential part of Smirnov's form factor formula is the pairing

$$\langle \ell, L \rangle_\alpha = \det\Big(\int_{-\infty}^{\infty} \chi(\sigma) e^{\alpha\sigma/p} \ell^{(i)}(\mathfrak{s}) L^{(j)}(S) d\sigma \Big)_{i,j=1,\dots,n}$$

where $\mathfrak{s} = e^{2\sigma/p}$, $S = e^{\sigma}$. Function $\chi(\sigma)$ is known and fixed. So is the polynomial ("cocycles")

$$\ell(\mathfrak{s}_1, \dots, \mathfrak{s}_n) = \ell^{(1)} \wedge \cdots \wedge \ell^{(n)} .$$

The freedom for choosing a local operator lies in the choice of a polynomial ("cycles")

$$\boldsymbol{L_n(S_1,\ldots,S_n|B_1,\ldots,B_{2n}) = L^{(1)}\wedge\cdots\wedge L^{(n)}}\,,$$

which is skew-symmetric in $\boldsymbol{S_i}$ and symmetric in $\boldsymbol{B_j = e^{\beta_j}}$.

Smirnov explained that one should view these integrals as a *deformation of abelian integrals* **on a hyperelliptic curve of genus $\boldsymbol{g = n-1}$ [29][31]. In particular there are deformed analogs of exact forms and the Riemann bilenear identity, which entail relations among these integrals.**

This point of view will play an important role in Smirnov's subsequent work.

Residue axiom (4) boils down to the recursion relations

$$(\sharp) \quad L_n(S_1, \ldots, S_{n-1}, B | B_1, \ldots, B_{2n-2}, B, -B)$$
$$= B \prod_{j=1}^{n-1} (B^2 - S_j^2) \cdot L_{n-1}(S_1, \ldots, S_{n-1} | B_1, \ldots, B_{2n-2}) \, .$$

So, in the bootstrap approach, a local operator is nothing but a sequences of polynomials (towers)

$$L_\star = \{L_n(S_1, \ldots, S_n | B_1, \ldots, B_{2n})\}_{n \geq 0} \quad \text{satisfying } (\sharp).$$

For special values of α, towers must be considered modulo null vectors, i.e. towers for which integrals vanish. The latter arise due to deformed exact forms and deformed Riemann bilinear identity.

This description of local fields is very different from the traditional picture.

The sG model is an integrable perturbation of a CFT (complex Liouville model). One expects that local fields in the former are in one-to-one correspondence with those in the ultraviolet limit to CFT.

How does the bootstrap picture match with CFT?

Smirnov [32] addressed the question of counting all local fields in sG model, showing that all 'minimal' form factors can be extended to towers.

Babelon-Bernard-Smirnov [38] introduced certain linear operators (BBS fermions) which act on the space of towers. They successfully described null vectors in the fermionic language, obtaining the correct character of the chiral CFT.

4. Fermionic structure: (a) Lattice models

Correlation functions in one-dimensional XXZ spin chains

$$H_{XXZ} = \sum_{j\in\mathbb{Z}} \Big(\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+ + \frac{q+q^{-1}}{4} \sigma_j^3 \sigma_{j+1}^3 \Big)$$

satisfy reduced qKZ equations at level -4. They are given by multiple integral formulas.

Miwa, Miki, Nakayashiki, J. 1992, 1996

Kitanine, Maillet, Terras 2000

These integrals are "computable" using 2 transcendental functions with very complicated rational coefficients

Boos, Korepin 2001

Sato, Shiroishi, Takahashi 2005

To understand this phenomenom, the issue was to find a way of organizing local operators on the lattice.

Smirnov and collaborators (Boos, Miwa, Takeyama, J.) introduced fermions which act on the space of *quasi-local* operators [62],[64]

$$\mathcal{W}_\alpha = \{q^{2\alpha S(0)}\mathcal{O} \mid \mathcal{O}\text{: local}\}\,, \quad S(0) = \frac{1}{2}\sum_{j=-\infty}^{0} \sigma_j^3\,.$$

The fermionic creation/annihilation operators, together with the adjoint action by local integrals of motion I_p, $p \in \mathbb{Z}_{>0}$

$$\mathrm{b}_p^*,\ \mathrm{c}_p:\ \mathcal{W}_{\alpha+1} \to \mathcal{W}_\alpha\,, \quad \mathrm{b}_p,\ \mathrm{c}_p^*:\ \mathcal{W}_{\alpha-1} \to \mathcal{W}_\alpha\,,$$

$$\mathrm{t}_p^* = [I_p, \cdot]:\ \mathcal{W}_\alpha \to \mathcal{W}_\alpha\,,$$

satisfy

$$[\mathrm{b}_p, \mathrm{b}_r^*]_+ = [\mathrm{c}_p, \mathrm{c}_r^*]_+ = \delta_{p,r}\,, \quad [\mathrm{t}_p^*, \text{anything}] = 0\,,$$

All other combinations (anti)commute.

They create a fermionic basis of quasi-local operators

$$q^{2\alpha S(0)}\mathcal{O} = \mathrm{t}_{p_1}^* \cdots \mathrm{t}_{p_m}^* \mathrm{b}_{r_1}^* \cdots \mathrm{b}_{r_n}^* \mathrm{c}_{s_1}^* \cdots \mathrm{c}_{s_n}^* \left(q^{2\alpha S(0)}\right)$$

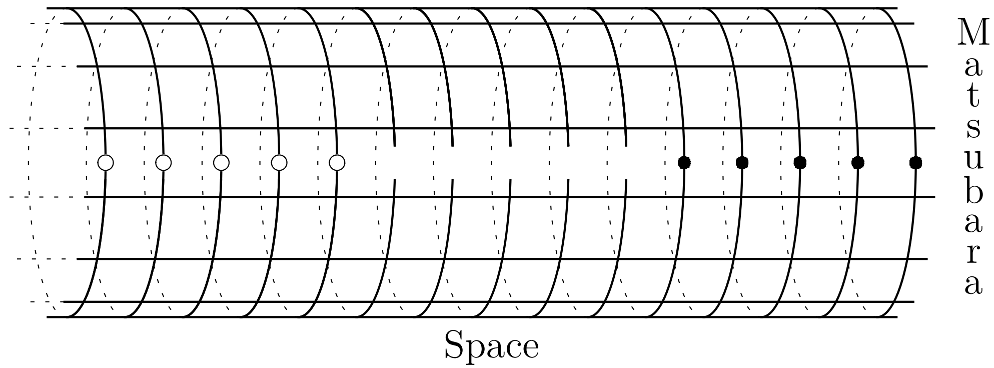
Construction of fermions hinges on the q-oscillator construction developed by Bazhanov-Lukyanov-Zaomolodchikov 1999.

The virtue of the fermionic basis is that their expectation values are simple.

Introduce expectation values on a cylinder

$$\mathcal{Z}_{\mathrm{n}}^{\kappa}\left\{q^{2\alpha S(0)}\mathcal{O}\right\} = \frac{\mathrm{Tr}_{\mathrm{S}}\mathrm{Tr}_{\mathrm{M}}\left(T_{\mathrm{S,M}}q^{2\kappa S+2\alpha S(0)}\mathcal{O}\right)}{\mathrm{Tr}_{\mathrm{S}}\mathrm{Tr}_{\mathrm{M}}\left(T_{\mathrm{S,M}}q^{2\kappa S+2\alpha S(0)}\right)}$$

Number of sites n and parameters (κ and the inhomogeneity parameters) in the Matsubara direction are arbitrary.



Main Theorem:[65] **There exist functions $\rho(\eta)$ and $\omega(\zeta,\xi)$, depending on α and the Matsubara data, such that**

$$\mathfrak{Z}_{\mathrm{n}}^{\kappa}\bigl\{\mathrm{t}^*(\eta_1)\cdots\mathrm{t}^*(\eta_m)\mathrm{b}^*(\zeta_1)\cdots\mathrm{b}^*(\zeta_n)\mathrm{c}^*(\xi_1)\cdots\mathrm{c}^*(\xi_n)(q^{2\alpha S(0)})\bigr\}$$
$$=\prod_{j=1}^{m}2\rho(\eta_j)\cdot\det\bigl(\omega(\zeta_i,\xi_j)\bigr)_{i,j=1,\dots,n}\,,$$

where

$$\mathrm{b}^*(\zeta)=\zeta^{2+\alpha+\mathbb{S}}\sum_{p=1}^{\infty}(\zeta^2-1)^{p-1}\mathrm{b}^*_p\,,\quad \mathrm{c}^*(\zeta)=\zeta^{2-\alpha-\mathbb{S}}\sum_{p=1}^{\infty}(\zeta^2-1)^{p-1}\mathrm{c}^*_p\,,$$
$$\mathrm{t}^*(\zeta)=\sum_{p=1}^{\infty}(\zeta^2-1)^{p-1}\mathrm{t}^*_p\,.$$

For the proof, the crucial idea is a *discrete version* **of deformed abelian integrals. The function $\omega(\xi,\zeta)$ is a quantum analog of the canonical differential of the second kind.**

Inverse problem: given a quasi-local operator, find its coefficients in the fermionic basis

$$q^{2\alpha S(0)}\mathcal{O} = \sum_{I^+,I^0,I^-} C_{I^+,I^0,I^-}\, \mathrm{b}^*_{I^+}\mathrm{t}^*_{I^0}\mathrm{c}^*_{I^-}\left(q^{2\alpha S(0)}\right),$$

$$\mathrm{b}^*_{I^+} = \mathrm{b}^*_{r_1}\cdots \mathrm{b}^*_{r_n} \quad \text{for } I^+ = \{r_1,\dots,r_n\}, \text{ etc.}$$

Smirnov showed that Main Theorem gives an efficient algorithm.

(i) Give Matsubara data *and Bethe roots*,

(ii) Solve for $\rho(\eta)$, $\omega(\xi,\zeta)$,

(iii) Insert these data into Main Theorem and solve for C_{I^+,I^0,I^-}.

Each step involves only *linear* **equations.**

Applications include

- **OPE coefficients and correlation functions (upto 11 sites) [80][81]**
- **density matrix and entanglememt entropy [84]**

5. Fermionic structure: (b) Field theory

By taking scaling limit of the functions $\rho(\eta)$, $\omega(\xi,\zeta)$, one can introduce fermionic basis of local operators in field theory. These fields are some descendants of the exponential field $e^{i\tilde{\alpha}\varphi(0)}$. The question is, *who are they?* To answer that, one has to study first the CFT limit and identify fermionic descendants with Virasoro descendants.

Scaling variables $\zeta = (Ca)^{\nu}\lambda$, $q = e^{\pi i\nu}$, and taking the limit $\mathrm{n} \to \infty$, lattice spacing $a \to 0$ keeping $\mathrm{n}a$ fixed, define

$$2\tau^*(\lambda) = \lim \mathrm{t}^*(\zeta)\,, \quad 2\beta^*(\lambda) = \lim \mathrm{b}^*(\zeta)\,, \quad 2\gamma^*(\lambda) = \lim \mathrm{c}^*(\zeta)\,.$$

The expectation value tends to three point functions in chiral CFT on the cylinder

$$\frac{\langle \Delta_+ | X_\Delta | \Delta_- \rangle}{\langle \Delta_+ | \phi_\Delta | \Delta_- \rangle}$$

where X_Δ is a fermionic descendant of a promary field $\phi_\Delta = \phi_{\Delta_\alpha}$ and primary fields $\phi_{\Delta_\pm}$are inserted at $\pm\infty$. (To take $\Delta_+ = \Delta_{\kappa+\alpha}$ and $\Delta_- =$

Δ_{κ} as independent, insertion of *fermionic screening operators* **is necessary.)**

Technically it is simpler to take $\Delta_+ = \Delta_-$, where the integrals of motion $\tau^*(\lambda)$ do not contribute. In [67], fermionic descendants are identified with the Viraoro descendants up to level 6, *modulo integals of motion***.**

Negro-Smirnov [73] explained the structure of these formulas from the point of view of relection relations. Their simplified method gave answers at level 10.

Incorporating integrals of motion demands further work which was carried out by Boos-Smirnov [82].

Scaling limit to sine-Gordon model [69]

For simpicity, consider the infinite volume limit $R \to \infty$.

(i) It turns out [71] that

$$\text{BBS fermions} = \text{BJMS fermions}$$

(ii) One point functions of the descendants [68]

$$\frac{\langle \bar{\beta}^*_{\bar{I}^+}\bar{\gamma}^*_{\bar{I}^-}\beta^*_{I^+}\gamma^*_{I^-}\Phi_\alpha\rangle}{\langle\Phi_\alpha\rangle} = \delta_{\bar{I}^-,I^+}\delta_{\bar{I}^+,I^-}(-1)^{|I^+|}\big(i(p+1)\mu^{2(p+1)}\big)^{|I^+|+|I^-|}$$
$$\times \prod_{m\in I^+}\cot\tfrac{\pi}{2}(m(p+1)+\alpha)\prod_{m\in I^-}\cot\tfrac{\pi}{2}(m(p+1)-\alpha)\,.$$

(iii) Shift of α

$$\frac{\langle\Phi_{\alpha+2p}\rangle}{\langle\Phi_\alpha\rangle} = \frac{\langle\beta^*_1\bar{\gamma}^*_1\Phi^{(1)}_\alpha\rangle}{\langle\Phi_\alpha\rangle}\,,$$
$$\Phi^{(1)}_\alpha = i\mu^2C_1(\alpha)\cot\tfrac{\pi(2-\alpha)}{2(p+1)}\cdot\bar{\beta}^*_{\text{screen},1}\gamma^*_{\text{screen},1}\Phi_\alpha\,.$$

Formulas (ii),(iii) are enough to recover the known results for one point functions of the exponential fields and their first descendant.

Lukyanov-Zamolodchikov 1997

Fateev, Fradkin, Lukyanov, Zamolodchikov and Zamolodchikov 1999

Summary. **The development we have seen so far is well represented by the title of the book**

Local Operators in Quantum Integrable Models. vol.I

M. Jimbo, T. Miwa and F. Smirnov, 256, AMS, 2021

Smirnov's work provides us with

- **(i) understanding the space of** *all local operators*,
- **(ii) the same fermionic structure prevailing models on the lattice and in the continuum,**
- **(iii) beautiful deformation of classical mathematics,**
- **(iv) theoretical as well as practical advances.**