Boundary overlaps for the open spin chains

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Integrable systems and field theory
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The XXZ spin-1/2 Heisenberg chain

1. Periodic chain.

Hamiltonian

\[ H_{\text{bulk}} = \sum_{m=1}^{L} \left( \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta \left( \sigma_m^z \sigma_{m+1}^z - 1 \right) \right) \]

\[ \Delta = \cosh \zeta \] - anisotropy

Periodic boundary conditions: \( \sigma_{L+1} = \sigma_1 \).

2. Open chain.

\[ H = \sum_{m=1}^{L-1} \left( \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta \left( \sigma_m^z \sigma_{m+1}^z - 1 \right) \right) + h_- \sigma_1^- + h_+ \sigma_L^+ \]

\( h_\pm \) - boundary fields.

We consider \( \Delta > 1 \) - massive antiferromagnetic regime, \( \Delta = \cosh \zeta \)
Form Factors

The main question: systematic computation of the form-factors in the thermodynamic limit from the Algebraic Bethe ansatz

Form factors: matrix elements of local fields, local spin operators $\sigma^a_m, a = x, y, z$

$|\Psi_g\rangle$ the ground state of the model $|\Psi_e\rangle$ - an excited state

The most basic form factors

$$|\mathcal{F}_a(\Psi_e)|^2 = \frac{\langle \Psi_g | \sigma^a_m | \Psi_e \rangle \langle \Psi_e | \sigma^a_m | \Psi_g \rangle}{\langle \Psi_g | \Psi_g \rangle \langle \Psi_e | \Psi_e \rangle}$$
**Boundary overlaps**

Quench: dynamics of a system after abrupt change of one parameter.

We change **one** boundary field $h_- \rightarrow \tilde{h}_-$. Local change, but can drastically modify the ground state (globally).

$|\Psi\rangle$ the ground state before the change of field $|\tilde{\Psi}\rangle$ - ground state after the change of field.

The most basic overlap (gives for example the dominant term for the Loschmidt echo): scalar product of ground states.

$$|\mathcal{F}|^2 = \frac{\langle \Psi | \tilde{\Psi} \rangle \langle \tilde{\Psi} | \Psi \rangle}{\langle \Psi | \Psi \rangle \langle \tilde{\Psi} | \tilde{\Psi} \rangle}$$
XXZ chain: Algebraic Bethe ansatz


\[ T_a(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \]

- Diagonal elements \(\rightarrow\) commuting conserved charges: transfer matrix

\[ \mathcal{T}(\lambda) = \text{tr}_a T_a(\lambda) = A(\lambda) + D(\lambda), \quad [\mathcal{T}(\lambda), \mathcal{T}(\mu)] = 0 \]

- Hamiltonian:

\[ H = c \frac{\partial}{\partial \lambda} \log \mathcal{T}(\lambda) \bigg|_{\lambda = \frac{i\zeta}{2}}, \quad [H, \mathcal{T}(\lambda)] = 0 \]

- Non-diagonal elements \(\rightarrow\) creation/annihilation operators.
Bethe states

Ferromagnetic state: \(|0\rangle = |\uparrow\uparrow \ldots \uparrow\rangle\), \(A(\lambda) \vert 0\rangle = a(\lambda) \vert 0\rangle\), \(D(\lambda) \vert 0\rangle = d(\lambda) \vert 0\rangle\).

Off-shell Bethe states: \(|\Psi(\{\lambda_1, \ldots, \lambda_N\})\rangle = B(\lambda_1) \ldots B(\lambda_N) \vert 0\rangle\).

For any Bethe state we define Baxter polynomial and exponential counting function

\[ q(\lambda) = \prod_{j=1}^{N} \sin(\lambda - \lambda_j), \quad a(\lambda) = \frac{a(\lambda) q(\lambda + i\zeta)}{d(\lambda) q(\lambda - i\zeta)}. \]

if the Bethe equations are satisfied (on-shell Bethe state)

\[ a(\lambda_j) + 1 = 0, \quad j = 1, \ldots N \]

then it is an eigenstate of the transfer matrix and the Hamiltonian

\[ \mathcal{T}(\mu) \vert \Psi(\{\lambda\})\rangle = \tau(\mu) \vert \Psi(\{\lambda\})\rangle, \quad \tau(\mu) = (a(\mu) + 1) \frac{q(\mu - i\zeta)}{q(\mu)}. \]
Open spin chain, Algebraic Bethe ansatz

Boundary matrices satisfying reflection equation (Cherednik 1984)

\[ R_{12}(\lambda - \mu) K_1(\lambda) R_{12}(\lambda + \mu) K_2(\mu) = K_2(\mu) R_{12}(\lambda + \mu) K_1(\lambda) R_{12}(\lambda - \mu). \]

We consider only diagonal solution: \( K(\lambda) = \begin{pmatrix} \sinh(\lambda + \xi - i\zeta/2) & 0 \\ 0 & \sinh(\xi - \lambda - i\zeta/2) \end{pmatrix}, \)

Algebraic Bethe Ansatz, Sklyanin 1988, Double row monodromy matrices:

\( T(\lambda) \)-usual monodromy matrix, \( \hat{T}(\lambda) = \sigma_0^y T^{t_0}(-\lambda) \sigma_0^y \) returned monodromy matrix.

\( \mathcal{U}_-(\lambda) = T(\lambda) K_-(\lambda) \hat{T}(\lambda) = \begin{pmatrix} \mathcal{A}_-(\lambda) & \mathcal{B}_-(\lambda) \\ \mathcal{C}_-(\lambda) & \mathcal{D}_-(\lambda) \end{pmatrix}, \)

\( \mathcal{U}^{t_0}_+(\lambda) = T^{t_0}(\lambda) K^{t_0}_+(\lambda) \hat{T}^{t_0}(-\lambda) = \begin{pmatrix} \mathcal{A}_+(\lambda) & \mathcal{C}_+(\lambda) \\ \mathcal{B}_+(\lambda) & \mathcal{D}_+(\lambda) \end{pmatrix}, \)
Algebraic Bethe Ansatz, open chain

1. Transfer matrix:

\[ \mathcal{T}(\lambda) = \text{tr}_0\{ K_+(\lambda) U_-(\lambda) \} = \text{tr}_0\{ K_-(\lambda) U_+(\lambda) \}. \]

\[ [\mathcal{T}(\lambda), \mathcal{T}(\mu)] = 0 \]

2. Hamiltonian:

\[ H = c \frac{d}{d\lambda} \mathcal{T}(\lambda) \bigg|_{\lambda = -i\zeta/2} + \text{constant}. \]

\[ h_\pm = -\sinh \zeta \coth \xi_\pm \]

3. Bethe states, Baxter polynomials:

\[ |\psi_+ (\{\lambda\})\rangle = \prod_{k=1}^{N} B_+ (\lambda_j)|0\rangle \text{, } \]

\[ Q(\lambda) = \prod_{j=1}^{N} \sin(\lambda - \lambda_j) \sin(\lambda + \lambda_j) \]

Note: operators \( B_+(\lambda) \) don’t depend on \( h_- \).
Bethe equations

Counting function

\[ \mathcal{A}(\lambda) = \frac{a(\lambda)d(-\lambda)\sin(\lambda + i\xi + i\zeta/2)\sin(\lambda + i\xi + i\zeta/2)}{d(\lambda)a(-\lambda)\sin(\lambda + i\xi - i\zeta/2)\sin(\lambda + i\xi - i\zeta/2)} Q(\lambda + i\zeta) Q(\lambda - i\zeta) \]

if the parameters \( \lambda \) satisfy the Bethe equations:

\[ \mathcal{A}(\lambda_j) = 1 \]

\( |\psi_+({\lambda})\rangle \) is an eigenstate of the transfer matrix \( \mathcal{T}(\mu) \):

\[ \mathcal{T}(\mu) |\psi_+({\lambda})\rangle = \tau(\mu, \{\lambda_j\}) |\psi_+({\lambda})\rangle, \]

\[ \tau(\mu) = \left( \mathcal{A}(\mu) \frac{\sin(2\mu + i\zeta)}{\sin(2\mu - i\zeta)} + 1 \right) \frac{Q(\mu - i\zeta)}{Q(\mu)}. \]
Scalar products and norms, periodic case

N. Slavnov, 1989: \( \{\lambda_1, \ldots \lambda_N\} \) - solution of Bethe equations, \( \{\mu_1, \ldots \mu_N\} \) - generic

\[
\langle \Psi(\{\mu\}) | \Psi(\{\lambda\}) \rangle = \frac{\prod_{k=1}^{N} q(\mu_k - i\zeta)}{\prod_{j>k} \sin(\lambda_j - \lambda_k) \sin(\mu_k - \mu_j)} \det M(\{\lambda\}|\{\mu\}),
\]

\[
M_{j,k}(\{\lambda\}|\{\mu\}) = a(\mu_k) t(\lambda_j - \mu_k) - t(\mu_k - \lambda_j), \quad t(\lambda) = \frac{i \sinh \zeta}{\sin \lambda \sin(\lambda - i\zeta)}.
\]

Norms of the on-shell Bethe states are given by the Gaudin formula

\[
\langle \Psi(\{\lambda\}) | \Psi(\{\lambda\}) \rangle = (-1)^N \frac{\prod_{j=1}^{N} q(\lambda_j - i\zeta)}{\prod_{j \neq k} \sin(\lambda_j - \lambda_k)} \det \mathcal{N}(\{\lambda\}),
\]

\[
\mathcal{N}_{j,k}(\{\lambda\}) = a'(\lambda_j) \delta_{j,k} - K(\lambda_j - \lambda_k), \quad K(\lambda) = t(\lambda) + t(-\lambda).
\]
Computation of determinants

N.K. Maillet Terras '99: quantum inverse problem, we know that the computation of form factors can be reduced to the scalar products.

\[
S(\{\lambda\}|\{\mu\}) \equiv \frac{\langle \Psi(\{\mu\}) | \Psi(\{\lambda\}) \rangle \langle \Psi(\{\lambda\}) | \Psi(\{\mu\}) \rangle}{\langle \Psi(\{\lambda\}) | \Psi(\{\lambda\}) \rangle \langle \Psi(\{\mu\}) | \Psi(\{\mu\}) \rangle} = \prod_{j=1}^{N} \frac{q_{\lambda}(\mu_j)q_{\mu}(\lambda_j)}{q_{\lambda}(\lambda_j)q_{\mu}(\mu_j)} \cdot \frac{\det \mathcal{M}(\{\lambda\}|\{\mu\}) \det \mathcal{M}(\{\mu\}|\{\lambda\})}{\det \mathcal{N}(\{\lambda\}) \det \mathcal{N}(\{\mu\})}.
\]

The main idea is extremely simple: we compute the following matrices from a system of linear equations

\[
F_{\lambda} = \mathcal{N}^{-1}(\{\lambda\}) \mathcal{M}(\{\lambda\}|\{\mu\}), \quad F_{\mu} = \mathcal{N}^{-1}(\{\mu\}) \mathcal{M}(\{\mu\}|\{\lambda\}),
\]

\[
a'_{\lambda}(\lambda_j)F_{\lambda,j,k} - \sum_{a=1}^{N} K(\lambda_j - \lambda_a)F_{\lambda,a,k} = a_{\lambda}(\mu_k)t(\lambda_j - \mu_k) - t(\mu_k - \lambda_j).
\]
We set

\[ a'_\lambda(\lambda_j)F_{\lambda,j,k} = G_\lambda(\lambda_j; \mu_k) \]

Linear equations \(\rightarrow\) Contour integral equation for a meromorphic function \(G_\lambda(\lambda; \mu)\)

\[
G_\lambda(\lambda; \mu_k) - \frac{1}{2\pi i} \oint_{\Gamma} d\nu \frac{K(\lambda - \nu)G_\lambda(\nu; \mu_k)}{1 + a_\lambda(\nu)} = (a_\lambda(\mu_k) + 1)t(\lambda - \mu_k),
\]
We set

$$G_\lambda(\lambda; \mu) = (1 + a_\lambda(\mu)) \rho_\lambda(\lambda; \mu)$$

Thermodynamic limit $\rightarrow$ Integral equation

$$\rho_\lambda(\lambda; \mu) + \frac{1}{2\pi\imath} \int_{-\pi/2+i0}^{\pi/2+i0} d\nu \, K(\lambda - \nu) \rho_\lambda(\nu; \mu) = t(\lambda - \mu).$$

**Lieb equation** for the density of Bethe roots! $\rightarrow$ **elliptic Cauchy determinant**

$$F_{\lambda, j, k} = \frac{a_\lambda(\mu_k) + 1}{a'_\lambda(\lambda_j)} \cdot \frac{(q^2, q^2)_\infty}{(-q^2, q^2)_\infty} \cdot \frac{\vartheta_2(\mu_k - \lambda_j, q)}{\vartheta_1(\mu_k - \lambda_j, q)}, \quad q = e^{-\zeta}$$
XXX case: 2-spinon form factor

N.K. G. Kulkarni '19: Matrix element of $\sigma_z$ between the ground state of the XXX chain and a state with 2 holes (spinons) $\mu_{h_1}$ and $\mu_{h_2}$

Final result for the form factor:

$$|\mathcal{Y}(\mu_{h_1} - \mu_{h_2})|^2 = \lim_{M \to \infty} M^2 |\mathcal{F}_z|^2 = \frac{2}{G^4 \left(\frac{1}{2}\right)} \left| \frac{G \left(\frac{\mu_{h_1} - \mu_{h_2}}{2i}\right) G \left(1 + \frac{\mu_{h_1} - \mu_{h_2}}{2i}\right)}{G \left(\frac{1}{2} + \frac{\mu_{h_1} - \mu_{h_2}}{2i}\right) G \left(\frac{3}{2} + \frac{\mu_{h_1} - \mu_{h_2}}{2i}\right)} \right|^2.$$  

Where $G(z)$ is the Barnes $G$-function (related to the double $\Gamma$-function).

$$G(z + 1) = \Gamma(z)G(z), \quad G(1) = 1.$$  

This reproduces the result for the two-spinon form factor obtained in the $q$-vertex operator framework from the M. Jimbo and T. Miwa multiple integral formulas by A. H. Bougourzi, M. Couture and M. Kacir.
Open chain: scalar products and norms

\[ S(\{\lambda\}|\{\mu\}) \equiv \frac{\langle \Psi(\{\mu\})|\Psi(\{\lambda\})\rangle \langle \Psi(\{\lambda\})|\Psi(\{\mu\})\rangle}{\langle \Psi(\{\lambda\})|\Psi(\{\lambda\})\rangle \langle \Psi(\{\mu\})|\Psi(\{\mu\})\rangle} \]

\[ = \prod_{j=1}^{N} \frac{Q_\lambda(\mu_j)Q_\mu(\lambda_j)}{Q_\lambda(\lambda_j)Q_\mu(\mu_j)} \cdot \frac{\det \mathcal{M}(\{\lambda\}|\{\mu\}) \det \mathcal{M}(\{\mu\}|\{\lambda\})}{\det \mathcal{N}(\{\lambda\}), \det \mathcal{N}(\{\mu\})}. \]

Slavnov matrix:

\[ \mathcal{M}_{j,k}(\{\lambda\}|\{\mu\}) = \mathfrak{A}_\lambda(\mu_k)t\left((-\mu_j + \lambda_j) - t(-\mu_k - \lambda_j)\right) + t(\mu_k - \lambda_j) - t(\mu_k + \lambda_j), \]

Gaudin matrix

\[ \mathcal{N}_{j,k}(\{\lambda\}) = \mathfrak{A}_\lambda'(\lambda_j)\delta_{j,k} - K(\lambda_j - \lambda_k) + K(\lambda_j + \lambda_k) \]
**Computation of determinants: open case**

Same idea as in the periodic case

\[
F_\lambda = \mathcal{N}^{-1}(\{\lambda\}) \mathcal{M}(\{\lambda\} | \{\mu\}), \quad F_\mu = \mathcal{N}^{-1}(\{\mu\}) \mathcal{M}(\{\mu\} | \{\lambda\}),
\]

Linear equations $\rightarrow$ Contour integral equation $\rightarrow$ Linear integral equation

\[
\rho_\lambda(\lambda; \mu) + \frac{1}{2\pi i} \int_{-\pi/2+i0}^{\pi/2+i0} d\nu \, K(\lambda - \nu) \rho_\lambda(\nu; \mu) = t(\lambda - \mu) + t(\lambda + \mu).
\]

Solution:

\[
F_{\lambda_j,k} = \frac{\mathcal{A}_\lambda(\mu_k) - 1}{\mathcal{A}_\lambda(\lambda_j)} \cdot \frac{(q^2, q^2)_\infty}{(-q^2, q^2)_\infty} \left( \frac{\vartheta_2(\lambda_j - \mu_k, q)}{\vartheta_1(\lambda_j - \mu_k, q)} + \frac{\vartheta_2(\mu_k + \lambda_j, q)}{\vartheta_1(\mu_k + \lambda_j, q)} \right)
\]

Once again **Cauchy determinant**
Cauchy determinant: open case

We use the following notations:

- ratio of the transfer matrix eigenvalues

\[ \chi(\lambda) = \frac{\tau(\lambda, \{\mu_j\})}{\tau(\lambda, \{\lambda_j\})} \]

- and the following function

\[ \varphi(\lambda, q) = \frac{\vartheta_1(\lambda, q)}{\sin \lambda} \]

Then we express the overlap as follows

\[ S(\{\lambda\} | \{\mu\}) = \prod_{j=1}^{N} \frac{\chi(\lambda_j)}{\chi(\mu_j)} \prod_{j,k=1}^{N} \frac{\varphi(\lambda_j - \lambda_k, q)\varphi(\mu_j - \mu_k, q)\varphi(\lambda_j + \lambda_k, q)\varphi(\mu_j + \mu_k, q)}{\varphi^2(\lambda_j - \mu_k)\varphi^2(\lambda_j + \mu_k)} \]

It remains to fix the two states and compute products in the thermodynamic limit.
**Ground states**

Configurations of the Bethe roots in the ground state depends on the boundary magnetic fields: \( h_- = -\sinh \zeta \coth \xi_- \) (first site) and \( h_+ = -\sinh \zeta \coth \xi_+ \) (last site). There are several cases leading to different structures of the ground state (S. Grijalva, J. Di Nardis, V. Terras '19).

We consider 3 most important situations. We limit our analysis to the case \( h_- > h_+ \).

- \( \Delta - 1 < h_- < \Delta + 1 \): All \( L/2 \) the roots are real distributed with a density given by the Lieb equation.
- \( 0 < h_- < \Delta - 1 \). \( L/2 - 1 \) real roots and a **boundary root** \( \lambda_{BR} = -i(\zeta/2 + \zeta_-) + O(L^{-\infty}) \)
- \( h_+ < \Delta - 1, \Delta + 1 < h_- \): \( L/2 - 1 \) real roots and a **boundary root** \( \lambda_{BR} \)

We change one field \( h_- \rightarrow \tilde{h}_-, \xi_- \rightarrow \tilde{\xi}_- \).
Final result: only real roots

Notations: \( q = e^{-\zeta}, \ p = e^{-2\xi}, \ \tilde{p} = e^{-2\tilde{\xi}}. \)

\[
S(\{\lambda\}|\{\mu\}) = \frac{F^2(q^4 p \tilde{p})}{F(q^4 p^2) F(q^4 \tilde{p}^2)}, \quad F(u) = \prod_{n=0}^{\infty} \frac{(uq^{4n+4}, q^4)}{(uq^{4n+2}, q^4)}.
\]
Final result: one boundary complex root

\[ S(\{\lambda\}|\{\mu\}) = \frac{F^2(p^{-1}\tilde{p}^{-1})}{F(p^{-2})F'(\tilde{p}^{-2})} \]
Conclusion and outlook

Advantages of the new approach:

- Explicit results, no Fredholm determinants.
- We know how to deal with complex roots.
- Possibility to apply in a systematic way for all the regimes of the XXZ chain, periodic case, open case etc.

Open problems:

- Can we apply this method far from the ground state?
- Impurities, non-local quenches?
Happy birthday, Fedor!

Федя, с Днем Рождения!