Hybrid integrable systems

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1. Hybrid classical-quantum systems.

- (M, ω) - symplectic manifold, the phase space of the system.

- We want to have a quantum system on the top of the classical system:

$$\begin{array}{ccl} V & \leftarrow & V_x = \pi^{-1}(x) \simeq \mathbb{C}^n \\ {\scriptstyle \pi}\downarrow & & \\ M & & \end{array}$$

Assume for simplicity that quantum system is finite dimensional

Hermitian structure on each fiber, smooth in x.

- Associate bundle

$$\begin{array}{c} End(V) \\ \downarrow \\ M \end{array}$$

Hermitian str. on V_x →

→ $*$ - structure on $End(V_x)$

- $A = \Gamma(M, \mathrm{End}(V))$ the space of (smooth) sections

  $s : M \to \mathrm{End}(V)$, $x \mapsto s(x)$, $\pi(s(x)) = x$

  pointwise multiplication

  $$(s_1 s_2)(x) = s_1(x) s_2(x)$$

  has a $*$-structure, $s(x) \mapsto s(x)^*$, i.e. A is a $*$-algebra

- $Z(A) = C(M)_{\mathbb{C}} \cdot I \subset A$ (I is a section $x \mapsto I_x$)

  has a natural Poisson structure

  $$\{z_1, z_2\} = \omega^{-1}(dz_1 \wedge dz_2)$$

  A is a module over the subalgebra $Z(A) \subset A$

- We want A to be a module over the Poisson algebra $Z(A)$:

$$\{z, s_1 s_2\} = \{z, s_1\} s_2 + s_1 \{z, s_2\}$$

$$\{z_1 \{z_2\, s\}\} = \{\{z_1, z_2\}, s\} + \{z_2, \{z_1, s\}\}$$

(R., Voronov, Weinstein)

$$\{z, s\} = (\omega^{-1})^{ij}\, \partial_i z\, \nabla_j s$$

where $\nabla_i s = \partial_i s + [A_i, s]$ the covariant derivative with respect to a flat connection $A = \sum_i A_i dx^i$

$$F = dA + \frac{1}{2}[A \wedge A] = 0$$

<u>Def</u>. A hybrid quantum system is a Hermitian vector bundle over (M, ω) with a (Hermitian) flat connection on it.

(Poisson Azumaya algebras)

2. Hybrid integrable systems

is a lift of a classical integrable system to V.

- Classical integrable system:

(a)
$$\begin{array}{c} M_{2n} \\ \pi \downarrow \\ B_n \end{array}$$
$\leftarrow$ generic fiber is Lagrangian

(b) $I_1, \dots, I_n \in C^\infty(M_{2n})$

s.t. $\{I_i, I_j\} = 0$, $\{I_i\}$ independent

$\pi(x) = (I_1(x), \dots, I_n(x))$, $B_n \subset \mathbb{R}^n$

Let $x(t_1, \dots, t_n)$ be the result of the multitime evolution:

$$\frac{\partial x(t)}{\partial t_j} = \omega^{-1}(d I(x(t)))$$

$x(t)$ lies in the same level surface of $\{I_i\}$ as $x(0)$

We want to lift $x(t)$ to $\begin{array}{c} V \\ \downarrow \\ M \end{array}$, i.e. we

need special sections

$$M_1, \dots, M_n : M \to \mathrm{End}(V)$$

s.t. the evolution of a section $\psi : M \to V$

is

$$\nabla_{t_j} \psi_t(x) = -M_j(x(t)) \psi_t(x), \qquad x = x(0)$$

This is why we impose

$$[\nabla_{t_j} + M_j(x(t)), \nabla_{t_k} + M_k(x(t))] = 0 \qquad (*)$$

or, equivalently:

$$\{I_i, M_j\} - \{I_j, M_i\} + [M_i, M_j] = 0$$

Because $(t_1, \dots, t_n)$ are (local) coordinates on level surfaces

$$M_c = \{ x \in M \mid I_i(x) = c_i \}$$

$(*)$ means $(M_1, \dots, M_n)$ is a flat deformation of the flat connection A on M_c for each c.

Back to (a):

<u>Def.</u> A <u>hybrid integrable system</u> is a deformation of the flat connection A on each fiber $\pi^{-1}(x)$.

$$\begin{array}{ll} V & \\ \downarrow & \text{with a flat} \\ M_{2n} & \text{connection } A \end{array}$$

$$\begin{array}{ll} M_{2n} & \leftarrow \text{generic} \\ \downarrow \pi & \quad \text{fiber is} \\ B_n & \quad \text{Lagrangian} \end{array}$$

3. The relation to deformation quantization

(a) A_0 - associative algebra, $Z(A_0) \subset A_0$ be its center.

$A_\hbar$ - flat deformation family of A_0 ($\varphi_\hbar : A_\hbar \xrightarrow{\sim} A_0$ as a (topological) vector space)

$$\varphi_\hbar^{-1}(a) *_\hbar \varphi_\hbar^{-1}(b) = ab - i\hbar\, m_1(a, b) + O(\hbar^2), \quad \hbar \to 0$$

Well known fact:

(i) $m_1(z, z') - m_1(z', z) = \{z, z'\}$ is a Poisson structure on $Z(A_0)$

(ii) $\{z, s\}$ is the action of $Z(A_0)$ by derivations on A_0

If we add
- A_0 is finite dimensional over $Z(A_0)$
- fibers are simple
- $Z(A_0)$ has trivial Poisson center

we have a Poisson Azumaya algebra.
(hybrid quantum system)

(b) Assume we have an integrable system on the family $A_\hbar$, i.e. $I_1^\hbar, \dots, I_n^\hbar \in A_0$ s.t.

$$\left[\varphi_\hbar^{-1}(I_j^\hbar), \varphi_\hbar^{-1}(I_k^\hbar)\right] = 0$$

If as $\hbar \to 0$ $\quad I_k^\hbar = I_j \cdot I - i\hbar M_j + O(\hbar^2)$

terms of order $\hbar$ give

$$\{I_i, M_j\} - \{M_i, I_j\} + [M_i, M_j] = 0$$

i.e. a hybrid integrable system on A_0

(Similar in general: $I_\hbar \subset A_\hbar$ maximal commutative)

Examples

(a) $A = \mathrm{Diff}_\hbar(Q) \otimes \mathrm{End}(V)$, $\mathrm{Diff}_\hbar(Q) = \{ P(-i\hbar\frac{\partial}{\partial q}, q) \}$

(i) Assume $\hat{H} = H_0(-i\hbar\frac{\partial}{\partial q}, q) - i\hbar\, M(-i\hbar\frac{\partial}{\partial q}, q) + O(\hbar^2)$

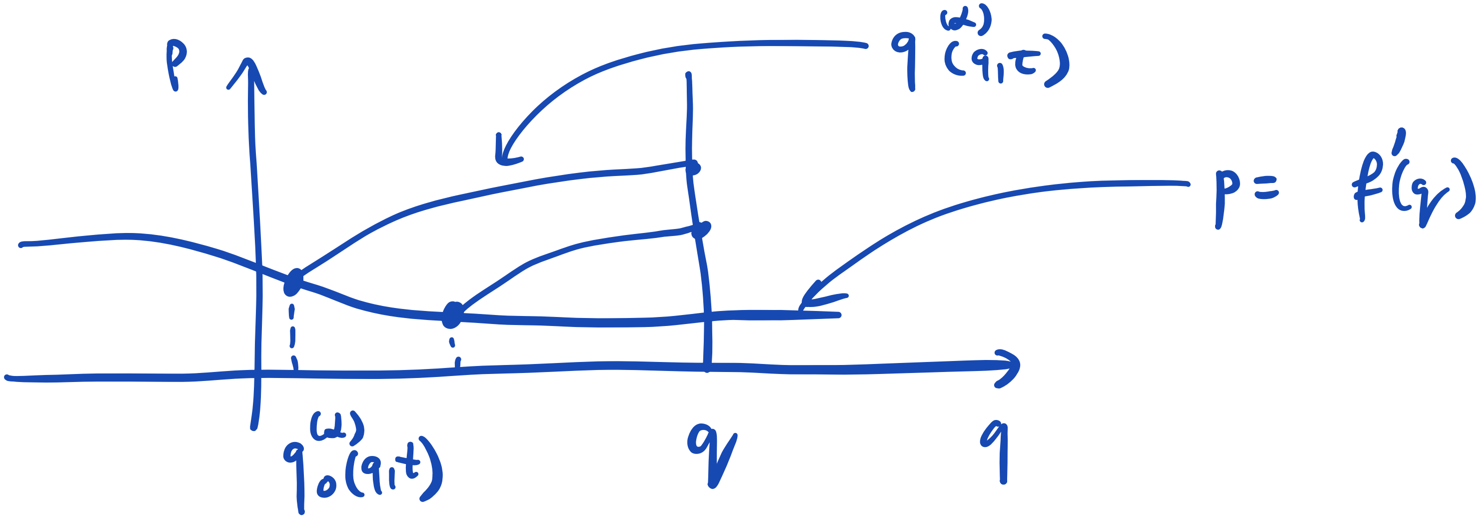
(symmetric)

The asymptotic $\hbar \to 0$ of $\psi(q,t)$

$$-i\hbar \frac{\partial \psi(q,t)}{\partial t} = \hat{H}\psi(q,t), \quad \psi(q,0) = e^{\frac{i f(q)}{\hbar}} \varphi(q)$$

Thm. $$\psi(q,t) = \sum_\alpha \left| \det\left(\frac{\partial q_0^{(\alpha)}(q,t)}{\partial q} \right) \right|^{\frac{1}{2}} e^{i\frac{S^{(\alpha)}(q,t)}{\hbar}} \psi^{(\alpha)}_{(q,t)} \left(1 + O(\hbar)\right)$$

where



$$S^{(\alpha)}(q,t) = \int_0^t \left(p^{(\alpha)}(\tau)\,\dot q^{(\alpha)}(\tau) - H_0\left(p^{(\alpha)}(\tau), q^{(\alpha)}(\tau)\right)\right) d\tau + f(q^{(\alpha)}(0))$$

$$\psi^{(\alpha)}(q,t) = \varphi^{(\alpha)}(t, q_0^{(\alpha)}(q,t))$$

where $\varphi^{(\alpha)}(t,q)$ is the solution to

$$\frac{\partial \varphi^{(\alpha)}(t,q_0)}{\partial t} = -i\, M\left(\frac{\partial S_t^{(\alpha)}}{\partial q}(q^{(\alpha)}(t,q_0)),\ q^{(\alpha)}(t,q_0)\right) \varphi^{(\alpha)}(t,q_0)$$

with initial condition

$$\varphi^{(\alpha)}(0,q_0) = \varphi(q_0)$$

where $q(t, q_0)$ is the trajectory with the initial condition

$$\left(\frac{\partial f}{\partial q}(q_0), q_0 \right)$$

(ii) If we have commuting integrals

$$[\hat{H}_j, \hat{H}_k] = 0$$

$$j, k = 1 \dots n$$

$$\hat{H}_j = H_j^{(cl)} \cdot I - i\hbar M_j + O(\hbar^2),$$

the semiclassical asymptotic of $\Psi(t_1, \dots, t_n)$ is similar

$$\frac{\partial \varphi^{(cl)}(t, q_0)}{\partial t_j} = -i M_j \left(\frac{\partial S}{\partial q}(q^{(cl)}(t, q_0)), q^{(cl)}(t, q_0) \right) \varphi^{(cl)}(t, q_0)$$

(iii) If x_* is a fixed point of the multitime evolution

$$[M_j(x_*), M_k(x_*)] = 0$$

we have a "quantum integrable system" on the fiber $\pi^{-1}(x_*)$.

(B) spin Calogero-Moser (Hikami-Wadati)

$$H_1 = \sum_{j=1}^{n} \left(-i\hbar \frac{\partial}{\partial q_j}\right),$$

$$H_2 = \sum_{j=1}^{n}\left(-\hbar^2 \frac{\partial^2}{\partial q_j^2}\right) + \sum_{i \neq j} \frac{(1+\hbar P_{ij})}{\sin^2(q_i - q_j)},$$

.....

$$\beta \to \infty \qquad H_2^{(sCM)} = H_2^{(CM)} + \hbar \sum_{i \neq j} \frac{P_{ij}}{\sin^2(q_i - q_j)} + \cdots$$

The hybrid integrable system

$$\left[i\frac{\partial}{\partial t_j} - M_j \,,\, \frac{\partial}{\partial t_k} - M_k \right] = 0$$

fixed point: $\quad q_j^* = \dfrac{2\pi j}{N} \,,$

$$M_j(q^*) = \text{Haldaine - Shastry model}$$

(c) Spin chains

Consider the Yangian type algebra $Y_\hbar$ with generators organized to families $\{T^U(u)\}$ where

$$R_{12}^{U,V}(u)\, T_1^U(u+v)\, T_2^V(v) = T_2^V(v)\, T_1^U(u+v)\, R_{12}^{U,V}(u)$$

where $R^{U,V}(u)$ is a family of R-matrices satisfying the Yang-Baxter equation and assume that they also form a semiclassical family

$$R^{U,V}(u) = I + i\hbar\, r^{U,V}(u) + O(\hbar^2)$$

Here $r^{U,V}(u)$ corresponding classical r-matrices satifying the classical Yang-Baxter equation.

Consider the representation of $Y_\hbar$ corresponding to a spin chain

$$T_a^U(u) = R_{a1}^{V,s_1}(u-v_1) \cdots R_{aN}^{V,s_N}(u-v_N)\, R_{a0}^{VU}(u-v)$$

Here U is the "auxiliary" space s_i are parameters of "spin" representations, $R^{V,s}(u)$ are corresponding R-matrices.

Corresponding transfer-matrices commute.

$$t^V(u) = \mathrm{Tr}_a\, T^V_a(u), \quad [t^V(u), t^U(v)] = 0$$

Now let us consider the semiclassical limit $\hbar \to 0$ such that $s_i = \frac{m_i}{\hbar}$, m_i are finite. In this limit

$$t^V(u) = t^V_{cl}(u) I_0 + i\hbar\, M^U_0(u,v)$$

Here I_0 is the identity operator acting in U,

$$t^V_{cl}(u) = \mathrm{tr}_a\left(L^{V,m_1}_{a1}(u-v_1) \ldots L^{V,m_N}_{aN}(u-v_N)\right)$$

is the classical transfer-matrix, $L_{a_i}^{V,m_i}(u-v_i)$ are Lax operators corresponding to i-th classical spin

$$\{L_a^{V,m}(u), L_b^{U,m}(v)\} = [r_{ab}^{V,U}(u-v), L_a^{V,m}(u) L_b^{U,m}(v)]$$

and

$$M_0^U(u) = tr_a\left(L_{a_1}^{V,m_1}(u-v_1)\cdots L_{aN}^{V,m_N}(u-v_N)\, r_{a0}^{V,U}(u-v)\right)$$

are corresponding M-operators.

Let $\{t_u\}$ be the time for the evolution $x(t_u)$ generated by $t(u)$:

$$\frac{\partial F_t}{\partial t_u} = \{t(u), F\}$$

Then we have well-known property of

commuting M-operators for the multitime evolution

$$\left[\frac{\partial}{\partial t_u} - M^U(u; x(t)),\ \frac{\partial}{\partial t_{u'}} - M^U(u'; x(t)) \right] = 0$$

Thus, in this case classical M-operators become the flat connection for the corresponding hybrid system.

At any fixed point of the multitime evolution we have commuting family

$$\left[M^U(u; x_*),\ M^U(u'; x_*) \right] = 0$$

The Azumaya Poisson structure in this case is trivial: trivial vector bundle and trivial flat connection A.

(d) Another example of the hybrid integrable system was obtained in 1995 together with Bazhanov & Bobenko. It is discrete S.G. model at roots of unity with discrete time diagonal evolution.

But that is a subject for a separate presentation.