Lattice realization of Generalized Affine Gaudin Model

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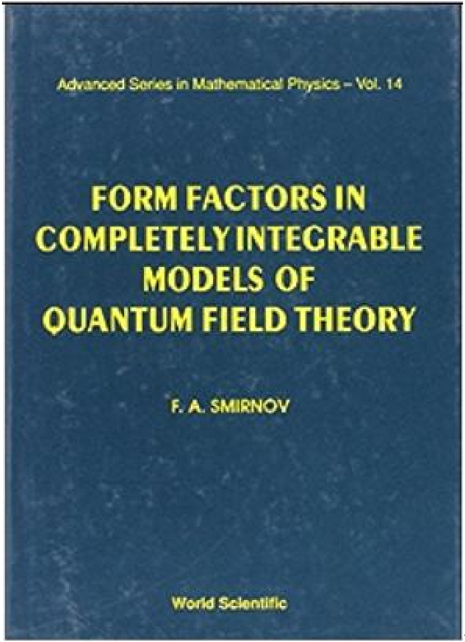
Fedya's Fest, October 12, 2023


Advanced Series in Mathematical Physics – Vol. 14
FORM FACTORS IN COMPLETELY INTEGRABLE MODELS OF QUANTUM FIELD THEORY
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;

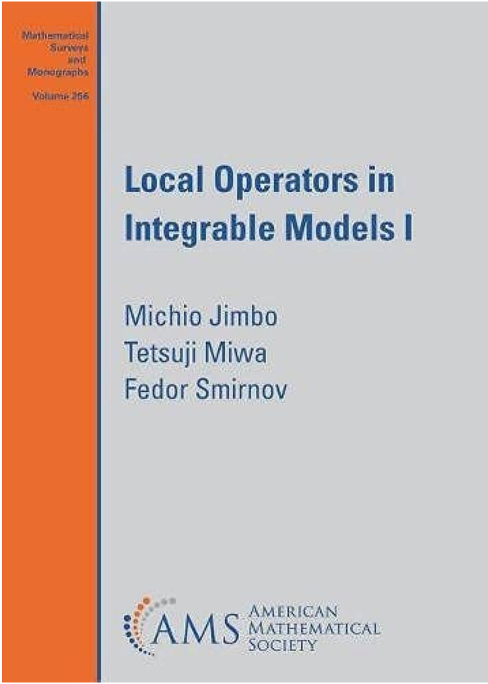
$$\times \prod_{p=1}^{k-2} \varphi_{m_p}^{-1}\left(\theta_{k-1}-\theta_p-\frac{i\xi m_k}{2}\right)\varphi_{m_p}^{-1}\left(\theta_{k-1}-\theta_p+\frac{i\xi m_k}{2}-\pi i\right)$$
$$\times \fint_{-\infty}^{\infty} d\alpha_1 \dots \fint_{-\infty}^{\infty} d\alpha_{n-1}\int d\alpha_n \dots \int d\alpha_{n+k-1}\prod \varphi(\alpha_i-\beta_j)$$
$$\times \prod_{p=1}^{k-2}\varphi_{m_p}(\alpha_i-\theta_p)\exp\Big(-\frac{\pi}{\xi}(n-2)\sum_{i=1}^{n-1}\alpha_i\Big)\prod_{i<j}\mathrm{sh}\,(\alpha_i-\alpha_j)$$
$$\times \prod_{i=1}^{n+k-1}\mathrm{sh}^{-1}\left(\alpha_i-\theta_{k-1}-\frac{i\xi m_k}{2}\right)\mathrm{sh}^{-1}\left(\alpha_i-\theta_{k-1}+\frac{i\xi m_k}{2}\right)$$
$$\times \langle F_n^{(1)}\rangle_n\left(\exp\frac{2\pi}{\xi}\alpha_1,\dots,\exp\frac{2\pi}{\xi}\alpha_{n-1}|\beta_1,\dots\beta_{2n}\right)\ , \qquad (83)$$

where we have used the identitites

$$\zeta_{s,m}(\beta)\zeta_{s,m}(\beta-\pi i)=\varphi^{-1}\left(-\beta-\frac{i\xi}{2}m\right)\varphi^{-1}\left(-\beta-\pi i+\frac{i\xi m}{2}\right)\ ,$$
$$\zeta_{m_1,m_2}(\beta)\zeta_{m_1,m_2}(\beta-\pi i)$$
$$=\varphi_{m_1}^{-1}\left(-\beta-\frac{i\xi m_2}{2}\right)\varphi_{m_1}^{-1}\left(-\beta-\pi i+\frac{i\xi m_2}{2}\right)\ ,$$
$$\varphi_m(\alpha)\varphi_m(\alpha-\pi i)=\mathrm{sh}^{-1}\left(\alpha-\frac{i\xi m}{2}\right)\mathrm{sh}^{-1}\left(\alpha+\frac{i\xi m}{2}\right)\ ,$$
$$2\pi i d_m^2\zeta_{m,m}(-\pi i)=\sin m\xi\ .$$

Consider the integrals over α_{n+k-1} and α_{n+k-2},

$$\frac{1}{2\pi i}\int_{\Gamma_{k-1}} d\alpha_{n+k-2}\prod_{j=1}^{2n}\varphi(\alpha_{n+k-2}-\beta_j)\prod_{p=1}^{k-2}\varphi_{m_p}(\alpha_{n+k-2}-\theta_p)$$
$$\times\prod_{i<n+k-2}\mathrm{sh}(\alpha_{n+k-2}-\alpha_i)$$
$$\times\frac{1}{\mathrm{sh}(\alpha_{n+k-2}-\theta_{k-1}-\frac{i\xi m_k}{2})\mathrm{sh}\,(\alpha_{n+k-2}-\theta_{k-1}+\frac{i\xi m_1}{2})}$$
$$\frac{1}{2\pi}\int_{\Gamma_k} d\alpha_{n+k-1}\prod_{j=1}^{2n}\varphi(\alpha_{n+k-1}-\beta_j)\prod_{p=1}^{k-2}\varphi_{m_p}(\alpha_{n+k-1}-\theta_p)$$
$$\times\prod_{i<n+k-1}\mathrm{sh}\,(\alpha_{n+k-1}-\alpha_i)$$
$$\times\frac{1}{\mathrm{sh}(\alpha_{n+k-1}-\theta_{k-1}-\frac{i\xi m_k}{2})\,\mathrm{sh}(\alpha_{n+k-1}-\theta_{k-1}+\frac{i\xi m_k}{2})}\ ,$$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The Algebraic Bethe Ansatz goes smoothly, the functions $a(\xi), d(\xi)$ change to

$$a(\xi)=\prod_{\mathbf{m}=1}^{\mathbf{n}}\left(1-q^{-2s_{\mathbf{m}}}(\xi/\tau_{\mathbf{m}})^2\right),\quad d(\xi)=\prod_{\mathbf{m}=1}^{\mathbf{n}}\left(1-q^{2s_{\mathbf{m}}}(\xi/\tau_{\mathbf{m}})^2\right). \tag{2.93}$$

At several places $\mathbf{n}$ from Section 2.1 changes to $\mathbf{N}$ which is the number of spin-1/2 representations before fusion. For given spin s we introduce the union of Bethe roots

$$\{\lambda_j\}_{j=1}^{\mathbf{N}}=\{\lambda_j^-(\kappa+\alpha)\}_{j=1}^{\frac{\mathbf{N}}{2}-s}\ \cup\ \{\lambda_j^+(\kappa)\}_{j=1}^{\frac{\mathbf{N}}{2}+s}.$$

Using

$$\mathcal{P}^{2s_2+1}_{2s_2,1}|\downarrow\rangle=|\downarrow\rangle,\quad \langle\uparrow|\mathcal{P}^{2s_2+1}_{2s_2,1}=\langle\uparrow|$$

one derives for the scalar product

(2.94)

$$\langle\uparrow|\prod_{j=1}^{\mathbf{N}}C(\lambda_j)|\downarrow\rangle=\prod_{j=1}^{\mathbf{N}}\prod_{\mathbf{m}=1}^{\mathbf{n}}P(\lambda_j/\tau_{\mathbf{m}},1/2,s_{\mathbf{m}})\ M_{\mathbf{N}}(\lambda_1,\cdots,\lambda_{\mathbf{N}}|\theta_1,\cdots\theta_{\mathbf{N}}).$$

Graphically the left hand side is represented as follows

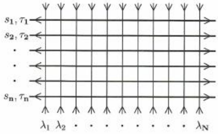


FIG. 7.

This does not correspond to the six-vertex model anymore: there are $2s_{\mathbf{m}}+1$ states on horizontal lines now. However, the boundary conditions still correspond to lowest and highest vectors, hence they are drawn as left and right arrows.

For higher spins the formula (2.94) gives an expression for the scalar product $\langle\kappa+\alpha|\kappa\rangle$ in terms of a huge determinant of the size $\mathbf{N}\times\mathbf{N}$. Later we shall see that a more economic formula exists, the determinant being of the size $\mathbf{n}\times\mathbf{n}$.

2.4.1. Summary. We consider the transfer-matrices with different auxiliary spaces introduced above. For quantum space we can take generalised *Matsubara* space (2.81). Another example, used in the next chapter, is the space

$$\mathrm{End}\big(V_{\xi_1}^{(1)}\big)\otimes\cdots\otimes\mathrm{End}\big(V_{\xi_n}^{(1)}\big) \tag{2.95}$$

on which the adjoint R-matrices act, see (3.11). The following theorem summarises the results of the present section. It holds irrespective of the nature of the quantum space (for example, (2.81) or (2.95)) which will be denoted by $\mathfrak{H}_{\mathrm{qtm}}$.

We shall denote by S_{qtm} the total spin in the quantum space. In (2.97) and (2.98) below, we use S_{qtm} in this sense. When we deal with R-matrices in adjoint action, S_{qtm} denotes its adjoint action.

Gaudin model [Gaudin'76]

- Quantum spins $\vec{S}^{(a)} = \big(S_1^{(a)}, S_2^{(a)}, S_3^{(a)}\big)$ $(a = 1, \ldots, r)$:

$$\big[S_A^{(a)}, S_B^{(b)}\big] = \mathrm{i}\,\delta_{ab}\ \varepsilon_{ABC}\ S_C^{(a)}\ , \qquad\qquad \big(\vec{S}^{(a)}\big)^2 = \mathrm{j}_a(\mathrm{j}_a + 1)$$

- Hamiltonians

$$\mathbf{H}^{(a)} = 2 \sum_{\substack{b=1 \\ b\neq a}}^{r} \frac{\vec{S}^{(a)} \cdot \vec{S}^{(b)}}{z_a - z_b} \ \ :$$

$$\sum_{a=1}^{r} \mathbf{H}^{(a)} = 0\ , \qquad [\mathbf{H}^{(a)}, \mathbf{H}^{(b)}] = 0$$

Bethe Anzatc Equations

- $$\mathbf{H}^{(a)} = 2 \sum_{\substack{b=1 \\ b\neq a}}^{r} \frac{\vec{S}^{(a)} \cdot \vec{S}^{(b)}}{z_a - z_b} \ , \qquad \qquad \big(\vec{S}^{(a)}\big)^2 = \mathtt{j}_a(\mathtt{j}_a + 1)$$

- $$\sum_{a=1}^{r} \frac{\mathtt{j}_a}{z_a - x_m^{(+)}} - \sum_{\substack{n=1 \\ n\neq m}}^{\mathtt{M}_+} \frac{1}{x_n^{(+)} - x_m^{(+)}} = 0 \qquad \qquad (m = 1, 2, \ldots, \mathtt{M}_+)$$

  $\mathtt{M}_+ = 0, 1, \ldots, 2\sum_{a=1}^{r} \mathtt{j}_a$.

- $$E_a = \sum_{\substack{b=1 \\ b\neq a}}^{r} \frac{2\mathtt{j}_a \mathtt{j}_b}{z_a - z_b} - \sum_{m=1}^{\mathtt{M}_+} \frac{2\mathtt{j}_a}{z_a - x_m^{(+)}}$$

Inhomogeneous $\mathfrak{sl}_2$ spin chain in the "classical" limit

Lax operator for the inhomogeneous $\mathfrak{sl}_2$-invariant spin chain [Kulish, Reshetikhin., Sklyanin'82]

$$\mathbf{L}_n(u) = 1 + \frac{\hbar}{u}\ \vec{S}_n \cdot \vec{\sigma} = \frac{1}{u}\begin{pmatrix} u + \hbar\, S_n^3 & \hbar\, S_n^- \\ \hbar\, S_n^+ & u - \hbar S^3 \end{pmatrix} \qquad \left(S_n^{\pm} = S_n^1 \pm \mathrm{i}\ S_n^2\right)$$

- Monodromy matrix

$$\mathbf{M}(u) = \mathbf{L}_1(u - z_1) \cdots \mathbf{L}_N(u - z_r) \qquad \text{satisfies the YB equation}$$

- Quantum determinant [Isergin, Korepin'81]

$$\det{}_\hbar \mathbf{M} = A(u - \tfrac{\hbar}{2})D(u + \tfrac{\hbar}{2}) - B(u - \tfrac{\hbar}{2})C(u + \tfrac{\hbar}{2}) \ \ : \ \ [\det{}_\hbar \mathbf{M}(u), \mathbf{M}(u')] = 0$$

- In the "classical limit"

$$\mathbf{M} = 1 + \hbar\,\mathbf{M}_1 + \hbar^2\,\mathbf{M}_2 + \ldots \qquad \Longrightarrow \qquad \mathrm{Tr}(\mathbf{M}) = 2 + \hbar^2\ \mathrm{Tr}\big(\mathbf{M}_2\big) + \ldots$$

$$\det{}_\hbar \mathbf{M} = 1 + \hbar^2\ \mathrm{Tr}\big(\mathbf{M}_2 - \tfrac{1}{2}\ \mathbf{M}_1^2\big) + \ldots$$

$$\boldsymbol{\tau}(u) \equiv \tfrac{1}{2}\ \mathrm{Tr}\big(\mathbf{M}_1^2\big) = \sum_{a=1}^{r} \frac{\mathbf{H}^{(a)}}{u - z_a} + \sum_{a=1}^{r} \frac{\mathrm{j}_a(\mathrm{j}_a + 1)}{(u - z_a)^2} \quad : \quad [\boldsymbol{\tau}(u), \boldsymbol{\tau}(u')] = 0$$

Bethe ansatz equation

$$\prod_{a=1}^{r} \frac{u_m - z_a - \hbar\, \mathrm{j}_a}{u_m - z_a + \hbar\, \mathrm{j}_a} = \prod_{\substack{n=1\\ n\neq m}}^{\mathtt{M}_+} \frac{u_m - u_n - \hbar}{u_m - u_n + \hbar} \qquad \xrightarrow{\hbar\to 0} \qquad \sum_{m=1}^{r} \frac{\mathrm{j}_a}{u_m - z_a} = \sum_{\substack{n=1\\ n\neq m}}^{\mathtt{M}_+} \frac{1}{u_m - u_n}$$

$$u_m \equiv x_m^{(+)}$$

Relation to ODE [Gaudin'76, Sklyanin'87, E.Frenkel'04,...]

- $$\big(-\partial_z^2 + t_0(z)\big)\,\Psi = 0 \qquad \text{with} \qquad t_0(z) = \sum_{a=1}^{r}\left(\frac{\mathrm{j}_a\,(\mathrm{j}_a+1)}{(z-z_a)^2} + \frac{E_a}{z-z_a}\right)$$

  The ODE possesses r regular singular points at $z = z_a$. If

  $$\sum_{a=1}^{r} E_a = 0$$

  then $z = \infty$ is also a regular singularity so that the ODE is a Fuchsian one.

- If the residues $\{E_a\}_{a=1}^{r}$ in the potential $t_0(z)$ coincide with the set of energies corresponding to some common eigenvector of the Hamiltonians $\mathbf{H}^{(a)}$, then all the singular points at $z = z_a$ turn out to be apparent
  (A singularity z_a is called apparent if the ratio of any two solutions of the ODE is single valued in the vicinity of that point)

  $$\Psi_+(z) = \frac{\prod_{m=1}^{\mathtt{M}_+}(z - x_m^{(+)})}{\prod_{a=1}^{r}(z-z_a)^{\mathrm{j}_a}}$$

  is a solution of the ODE.

- $$\Psi_+(z) = \frac{\prod_{m=1}^{\mathtt{M}_+}(z - x_m^{(+)})}{\prod_{a=1}^{r}(z - z_a)^{\mathrm{j}_a}}$$

$$\sum_{a=1}^{r} \frac{\mathrm{j}_a}{z_a - x_m^{(+)}} - \sum_{\substack{n=1 \\ n \neq m}}^{\mathtt{M}_+} \frac{1}{x_n^{(+)} - x_m^{(+)}} = 0 \qquad (m = 1, 2, \ldots, \mathtt{M}_+)$$

- There is another linearly independent solution of the form

$$\Psi_-(z) = z\, \frac{\prod_{m=1}^{\mathtt{M}_-}(z - x_m^{(-)})}{\prod_{a=1}^{r}(z - z_a)^{\mathrm{j}_a}} \qquad \Big(\mathtt{M}_+ + \mathtt{M}_- = 2\sum_{a=1}^{r} \mathrm{j}_a\Big).$$

Here the set $\{x_m^{(-)}\}_{m=1}^{\mathtt{M}_-}$ also solves the Bethe ansatz like equations,

$$\frac{1}{x_m^{(-)}} + \sum_{a=1}^{r} \frac{\mathrm{j}_a}{z_a - x_m^{(-)}} - \sum_{\substack{n=1 \\ n \neq m}}^{\mathtt{M}_-} \frac{1}{x_n^{(-)} - x_m^{(-)}} = 0 \qquad (m = 1, 2, \ldots, \mathtt{M}_-)\,,$$

$$\begin{aligned} E_a \;&=\; \sum_{\substack{b=1 \\ b \neq a}}^{r} \frac{2\mathrm{j}_a \mathrm{j}_b}{z_a - z_b} - \sum_{m=1}^{\mathtt{M}_+} \frac{2\mathrm{j}_a}{z_a - x_m^{(+)}} \\ &-\; \frac{2\mathrm{j}_a}{z_a} + \sum_{\substack{b=1 \\ b \neq a}}^{r} \frac{2\mathrm{j}_a \mathrm{j}_b}{z_a - z_b} - \sum_{m=1}^{\mathtt{M}_-} \frac{2\mathrm{j}_a}{z_a - x_m^{(-)}} \end{aligned}$$

ODE/IQFT correspondence

There is a link between the spectrum of the Gaudin Hamiltonians and a class of differential equations possessing certain monodromy properties. This provides one of the simplest illustrations of a broad phenomena, known as the ODE/IQFT correspondence

[Voros'94; Dorey,Tateo'98; Bazhanov,Lukyanov,Zamolodchikov'98;...]

- The Gaudin model admits a generalization to any simple Lie algebra $\mathfrak{g}$ [Gaudin'83;Jurčo'89; Feigin,Frenkel,Reshetikhin'94,...]

- The development of the mathematical apparatus of $2D$ CFT led to the idea that there should be a meaningful generalization to the case when the finite-dimensional Lie algebra is replaced by an Kac-Moody algebra $\widehat{\mathfrak{g}}$.

- The diagonalization problem would be formulated for an infinite set $\{\mathbb{I}_s\}$ of local IM which depend on the arbitrary parameters $\{z_a\}_{a=1}^r$:

$$\mathbb{I}_s = \int_0^{2\pi} \frac{\mathrm{d}u}{2\pi}\ T_{s+1}(u)\ \ : \qquad [\mathbb{I}_s, \mathbb{I}_{s'}] = 0$$

T_{s+1} is a chiral local field of Lorentz spin $s+1$.

Affine Gaudin model [Feigin, Frenkel'07]

- Consider r independent copies of the Kac-Moody $\widehat{\mathfrak{sl}}_{k_a}(2)$ algebra at levels $k_a = 1, 2, \ldots$.

$$J_A^{(a)}(u) J_B^{(b)}(0) = -\delta_{ab} \left(\frac{k_a}{2u^2}\, \eta_{AB} + \frac{\mathrm{i}}{u}\, f_{AB}{}^C\, J_C^{(a)} \right) + O(1)\ .$$

- To each copy one can associate the Virasoro field

$$G^{(a)} = \frac{\eta^{AB} J_A^{(a)} J_B^{(a)}}{k_a + 2} = \frac{1}{4\,(k_a+2)}\, \left(J_0^{(a)} J_0^{(a)} + 2\, J_+^{(a)} J_-^{(a)} + 2\, J_-^{(a)} J_+^{(a)} \right)$$

- Hamiltonians

$$\mathbf{H}_{\mathrm{G}}^{(a)} = \frac{1}{2} \int_0^{2\pi} \frac{\mathrm{d}u}{2\pi} \sum_{\substack{b=1 \\ b\neq a}}^{r} \frac{k_b\, G^{(a)} + k_a\, G^{(b)} - 2\eta^{AB} J_A^{(a)} J_B^{(b)}}{z_a - z_b}$$

- Conjecture [FF'07]: $\mathbf{H}_{\mathrm{G}}^{(a)} \in \{\mathbb{I}_s\} \longleftarrow$ infinite set of local IM

- **How to derive the Bethe Anzatc equations?**

Affine Gaudin model [Feigin, Frenkel'07]

Feigin and Frenkel put forward the conjecture, that the spectrum of

$$\mathbf{H}^{(a)}_{\mathrm{G}} = \frac{1}{2}\int_0^{2\pi}\frac{\mathrm{d}u}{2\pi}\sum_{\substack{b=1\\ b\neq a}}^{r}\frac{k_b\,G^{(a)}+k_a\,G^{(b)}-2\eta^{AB}J^{(a)}_A J^{(b)}_B}{z_a-z_b}$$

with

$$J^{(a)}_A(u)J^{(b)}_B(0) = -\delta_{ab}\left(\frac{k_a}{2u^2}\,\eta_{AB}+\frac{\mathrm{i}}{u}\,f_{AB}{}^{C}\,J^{(a)}_C\right)+O(1)$$

$$G^{(a)} = \frac{\eta^{AB}J^{(a)}_A J^{(a)}_B}{k_a+2}\ ,$$

would be encoded in a class of differential equations though they did not explain exactly how the spectrum would be extracted from a certain ODE.

Generalized affine $\mathfrak{sl}(2)$ Gaudin model [Kotousov,Lukyanov'21]

- $$\mathbf{H}^{(a)}_{\rm gen} = \int_0^{2\pi} \frac{\mathrm{d}u}{2\pi} \bigg[\frac{\beta^2 K}{1-\beta^2}\ G^{(a)} + \frac{1}{4K}\ \frac{1-\beta}{1+\beta}\ \Big(k_a \big(J_0^{(\rm tot)}\big)^2 - K\, J_0^{(a)} J_0^{(\rm tot)} \Big)$$

  $$-\sum_{\substack{b=1\\ b\neq a}}^{r} \frac{1}{z_a - z_b} \Big(\frac{1}{4}\,(z_a+z_b)\, J_0^{(a)} J_0^{(b)} + z_a\, J_+^{(b)} J_-^{(a)} + z_b\, J_+^{(a)} J_-^{(b)} - k_a z_b\, G^{(b)} - k_b z_a\, G^{(a)} \Big) \bigg]$$

  where

  $$J_0^{(\rm tot)} = \sum_{a=1}^{r} J_0^{(a)} \qquad \text{and} \qquad K = \sum_{a=1}^{r} k_a$$

- $\mathbf{H}^{(a)}_{\rm gen}$ depend on the parameter β. The Hamiltonians of the affine Gaudin model are obtained through a certain limiting procedure, which includes taking $\beta \to 1^-$.
- The ODE/IQFT correspondence for the model
- The spectrum of $\mathbf{H}^{(a)}_{\rm gen}$ for arbitrary $\beta \in (0,1)$

$$\mathbf{H}^{(a)}_{\rm gen} = \int_0^{2\pi} \frac{{\rm d}u}{2\pi} \Bigg[\frac{\beta^2 K}{1-\beta^2}\ G^{(a)} + \frac{1}{4K}\ \frac{1-\beta}{1+\beta}\ \Big(k_a\, \big(J_0^{(\rm tot)}\big)^2 - K\, J_0^{(a)} J_0^{(\rm tot)} \Big)$$

$$- \sum_{\substack{b=1 \\ b\neq a}}^{r} \frac{1}{z_a - z_b} \Big(\tfrac{1}{4}\,(z_a+z_b)\, J_0^{(a)} J_0^{(b)} + z_a\, J_+^{(b)} J_-^{(a)} + z_b\, J_+^{(a)} J_-^{(b)} - k_a z_b\, G^{(b)} - k_b z_a\, G^{(a)} \Big) \Bigg]$$

Gaudin Limit:

$$\beta \to 1 - \epsilon$$

$$z_a \to \tfrac{1}{\epsilon} + \epsilon z_a \quad \text{with } \epsilon \to 0$$

Affine Gaudin model Hamiltonians [Feigin, Frenkel'07]:

$$\mathbf{H}^{(a)} = \frac{1}{2} \int_0^{2\pi} \frac{{\rm d}u}{2\pi}\ \sum_{\substack{b=1 \\ b\neq a}}^{r} \frac{k_b\, G^{(a)} + k_a\, G^{(b)} - 2\eta^{AB} J_A^{(a)} J_B^{(b)}}{z_a - z_b}$$

ODE for GAGM

$$\left(-\partial_z^2 + t_{\mathrm{L}}(z) + \kappa^2\, \mathcal{P}(z) \right)\ \Psi = 0$$

where

$$t_{\mathrm{L}}(z) = -\frac{A^2+\frac{1}{4}}{z^2} + \sum_{a=1}^{r}\left(\frac{\mathrm{j}_a(\mathrm{j}_a+1)}{(z-z_a)^2} + \frac{z_a\gamma_a}{z(z-z_a)}\right) + \sum_{\alpha=1}^{\mathrm{L}}\left(\frac{2}{(z-w_\alpha)^2} + \frac{w_\alpha \Gamma_\alpha}{z(z-w_\alpha)}\right)$$

$$\mathcal{P}(z) = z^{-2+\xi\sum_{a=1}^{r} k_a}\ \prod_{a=1}^{r}(z-z_a)^{k_a} \qquad \left(\xi = \tfrac{\beta^2}{1-\beta^2}\right)$$

BA equations = a set of conditions that all the singularities are apparent except for $z=0$ and $z=\infty$.

Spectrum of $\mathbf{H}^{(a)}_{\rm gen}$ in GAGM

Eigenvalues of the GAGM Hamiltonians:

$$E^{(a)} = \frac{2z_a\gamma_a}{k_a} - 2z_a \sum_{\beta=1}^{\mathtt{L}} \frac{1}{z_a - w_\beta} - \frac{2z_a}{k_a} \sum_{\substack{b=1\\ b\neq a}}^{r} \frac{d_a k_b + d_b k_a}{z_a - z_b} - \frac{2d_a}{k_a}\,\big(\xi K - 2\big) - 2d_0$$

Here

$$d_a = \frac{\mathfrak{j}_\mathfrak{a}(\mathfrak{j}_\mathfrak{a}+1)}{k_a+2} - \frac{1}{24}\ \frac{3k_a}{k_a+2}\ , \qquad\qquad K = \sum_{a=1}^{r} k_a\ ,$$

while

$$d_0 = -\frac{1}{8} - \mathtt{L} - \sum_{a=1}^{r} d_a + \frac{1}{(1+\xi)K}\ \Big[2\mathtt{L} - A^2 + \sum_{a=1}^{r}\big(\mathfrak{j}_\mathfrak{a}(\mathfrak{j}_\mathfrak{a}+1) + z_a\gamma_a\big) + \sum_{\alpha=1}^{\mathtt{L}} w_\alpha \Gamma_\alpha\Big]$$

Lattice realization of the Generalized Affine Gaudin Model?

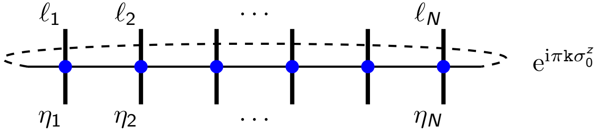
(in the spirit of the original Gaudin construction)

Inhomogeneous XXZ higher-spin

[Baxter'71,Kulish,Reshetikhin,Sklyanin'81,...]

$$\mathbb{T}(\zeta) = \mathrm{Tr}\left(\mathrm{e}^{\mathrm{i}\pi\mathrm{k}\sigma_0^z}\,\mathbf{L}_1^{(\ell_1)}(\zeta/\eta_1)\,\mathbf{L}_2^{(\ell_2)}(\zeta/\eta_2)\,\ldots\,\mathbf{L}_N^{(\ell_N)}(\zeta/\eta_N)\right)$$

$\mathbb{T}(\zeta) =$



$$\mathbf{L}^{(\ell)}(\zeta) = \begin{pmatrix} q^{\frac{1}{2}(1+H^{(\ell)})} + q^{\frac{1}{2}(1-H^{(\ell)})}\,\zeta & -(q-q^{-1})\,q\,\zeta\,F^{(\ell)} \\ (q-q^{-1})\,E^{(\ell)} & q^{\frac{1}{2}(1-H^{(\ell)})} + q^{\frac{1}{2}(1+H^{(\ell)})}\,\zeta \end{pmatrix}$$

Multiparametric integrable statistical system

$$\mathbb{T}(\zeta)\ : \qquad \left[\mathbb{T}(\zeta), \mathbb{T}(\zeta')\right] = 0$$

The Algebraic Bethe Ansatz goes smoothly, the functions $a(\xi), d(\xi)$ change to

$$a(\xi) = \prod_{\mathbf{m}=1}^{\mathbf{n}} \left(1 - q^{-2s_{\mathbf{m}}}(\xi/\tau_{\mathbf{m}})^2\right), \quad d(\xi) = \prod_{\mathbf{m}=1}^{\mathbf{n}} \left(1 - q^{2s_{\mathbf{m}}}(\xi/\tau_{\mathbf{m}})^2\right). \tag{2.93}$$

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$$\{\lambda_j\}_{j=1}^{\mathbf{N}} = \{\lambda_j^-(\kappa+\alpha)\}_{j=1}^{\frac{\mathbf{N}}{2}-s} \cup \{\lambda_j^+(\kappa)\}_{j=1}^{\frac{\mathbf{N}}{2}+s}.$$

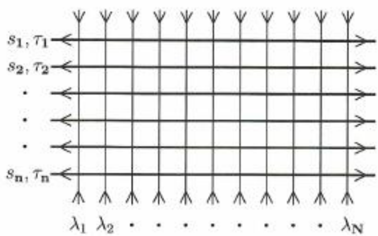
Using

$$\mathcal{P}_{2s_2,1}^{2s_2+1}|\downarrow\rangle = |\downarrow\rangle, \quad \langle\uparrow|\mathcal{P}_{2s_2,1}^{2s_2+1} = \langle\uparrow|$$

one derives for the scalar product

$$\langle\uparrow|\prod_{j=1}^{\mathbf{N}} C(\lambda_j)|\downarrow\rangle = \prod_{j=1}^{\mathbf{N}}\prod_{\mathbf{m}=1}^{\mathbf{n}} P(\lambda_j/\tau_{\mathbf{m}}, 1/2, s_{\mathbf{m}})\, M_{\mathbf{N}}(\lambda_1, \cdots, \lambda_{\mathbf{N}}|\theta_1, \cdots \theta_{\mathbf{N}}). \tag{2.94}$$

Graphically the left hand side is represented as follows



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Translational invariance

Translational invariance is a fundamental symmetry of a local QFT

$$\mathbb{T}(\zeta) = \mathrm{Tr}\left(\mathrm{e}^{\mathrm{i}\pi \mathrm{k}\sigma_0^z}\,\mathbf{L}_1^{(\ell_1)}(\zeta/\eta_1)\,\mathbf{L}_2^{(\ell_2)}(\zeta/\eta_2)\,\ldots\,\mathbf{L}_N^{(\ell_N)}(\zeta/\eta_N)\right)$$

Impose r-site periodicity condition:

$$\eta_{m+r} = \eta_m\ , \qquad \ell_{m+r} = \ell_m$$

with $N/r-$ an integer.

"Local" Hamiltonians

$$\mathbb{H}^{(a)} = 2\mathrm{i}\zeta\,\partial_\zeta \log\left(\mathbb{T}(-q^{-1}\,\zeta/\eta_a)\right)\Big|_{\zeta=1}\ : \qquad [\mathbb{H}^{(a)}, \mathbb{H}^{(b)}] = 0$$

For what values of the parameters is the model critical and what are the universality classes?

Scaling limit

- For the definition of the scaling limit, a central rôle belongs to

$$\mathbb{H} = \sum_{a=1}^{r} \mathbb{H}^{(a)} ,$$

which is essentially the logarithmic derivative of a r row transfer-matrix. The ground state of this Hamiltonian serves as the reference state from which the energy of the excited states is counted. In performing the scaling limit one takes $N \to \infty$ but considers only the class of states, whose excitation energy over the ground state energy is sufficiently low. Then as $N \to \infty$, the low energy spectrum organizes into the conformal towers, which are classified w.r.t to the algebra of (extended) conformal symmetry:

$$\mathbb{H} \asymp N e_\infty \mathbf{1} + \frac{2\pi r}{N} \, v_{\rm F} \left(\mathbf{I}_1 + \bar{\mathbf{I}}_1 \right) + o(N^{-1}) .$$

Here

$$\mathbf{I}_1 = \int_0^{2\pi} \frac{\mathrm{d}u}{2\pi} \; T(u) , \qquad \bar{\mathbf{I}}_1 = \int_0^{2\pi} \frac{\mathrm{d}u}{2\pi} \; T(u) ,$$

Scaling limit [Kotousov, SL'21]

$$\mathbb{H}^{(a)} \asymp N e_\infty^{(a)}\, \mathbf{1} + \frac{2\pi r}{N} v_{\rm F} \Big(\mathbf{I}_1^{(a)} + \bar{\mathbf{I}}_1^{(a)} \Big) + o(N^{-1})\,; \quad \mathbf{I}_1^{(a)} = \sum_b C_b^a\, \mathbf{H}_{\rm gen}^{(b)}\,, \quad \bar{\mathbf{I}}_1^{(a)} = \ldots$$

$$\mathbf{H}_{\rm gen}^{(a)} = \int_0^{2\pi} \frac{{\rm d}u}{2\pi} \Bigg[\frac{\beta^2 K}{1-\beta^2}\, G^{(a)} + \frac{1}{4K}\, \frac{1-\beta}{1+\beta} \Big(k_a \big(J_0^{(\rm tot)}\big)^2 - K\, J_0^{(a)} J_0^{(\rm tot)} \Big)$$

$$- \sum_{\substack{b=1 \\ b\neq a}}^r \frac{1}{z_a - z_b} \Big(\frac{1}{4}\,(z_a+z_b)\, J_0^{(a)} J_0^{(b)} + z_a\, J_+^{(b)} J_-^{(a)} + z_b\, J_+^{(a)} J_-^{(b)} - k_a z_b\, G^{(b)} - k_b z_a\, G^{(a)} \Big) \Bigg]$$

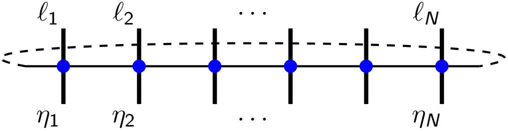
where

$$J_0^{(\rm tot)} = \sum_{a=1}^r J_0^{(a)} \qquad \text{and} \qquad K = \sum_{a=1}^r k_a$$

Identification of the parameters

$$\mathbf{L}^{(\ell)}(\zeta) = \begin{pmatrix} q^{\frac{1}{2}(1+H^{(\ell)})} + q^{\frac{1}{2}(1-H^{(\ell)})}\,\zeta & -(q-q^{-1})\,q\,\zeta\,F^{(\ell)} \\ (q-q^{-1})\,E^{(\ell)} & q^{\frac{1}{2}(1-H^{(\ell)})} + q^{\frac{1}{2}(1+H^{(\ell)})}\,\zeta \end{pmatrix}$$

$\mathbb{T}(\zeta) =$



$\mathrm{e}^{\mathrm{i}\pi\mathrm{k}\sigma_0^z}$

(a) $\eta_{a+r} = \eta_a \quad \text{with} \quad N = r\,L \to \infty$

(b) $k_a = 2\ell_a \quad \text{levels of } \widehat{\mathfrak{sl}}_{k_a}(2) \quad (a = 1, \ldots, r)$

(c) $q = -\mathrm{e}^{\frac{\mathrm{i}\pi}{K}(\beta^2-1)} \qquad \big(K = \sum_{a=1}^r k_a\big)$

(d) $\{\eta_a\}_{a=1}^r \mapsto \{z_a\}_{a=1}^r$

Spin chain for $r = 2$, $\ell_1 = \ell_2 = \frac{1}{2}$

$$q = \mathrm{e}^{\mathrm{i}\gamma} \,, \quad \eta_1 = (\eta_2)^{-1} = \mathrm{e}^{\mathrm{i}\alpha}$$

$$\mathbb{H}\big|_{r=2} = \frac{\cos(\gamma)}{2\sin(\gamma-\alpha)\,\sin(\gamma+\alpha)} \sum_{m=1}^{N} \Bigg[\frac{\sin^2(\alpha)}{\sin(\gamma)} \left(\sigma^x_m\,\sigma^x_{m+2} + \sigma^y_m\,\sigma^y_{m+2} + \sigma^z_m\,\sigma^z_{m+2} - \hat{\mathbf{1}} \right)$$

$$-2\sin(\gamma)\left(\frac{\cos^2(\alpha)}{\cos(\gamma)}\,(\sigma^x_m\,\sigma^x_{m+1} + \sigma^y_m\,\sigma^y_{m+1}) + \sigma^z_m\,\sigma^z_{m+1} - \hat{\mathbf{1}}\right) - \mathrm{i}\sin^2(\alpha)\,(\sigma^z_{m+2} - \sigma^z_{m-1})$$

$$\times(\sigma^x_m\sigma^x_{m+1} + \sigma^y_m\sigma^y_{m+1}) + (-1)^m\sin(2\alpha)\,\Big(\tan(\gamma)\,(\sigma^x_m\sigma^y_{m+1} - \sigma^y_m\sigma^x_{m+1})$$

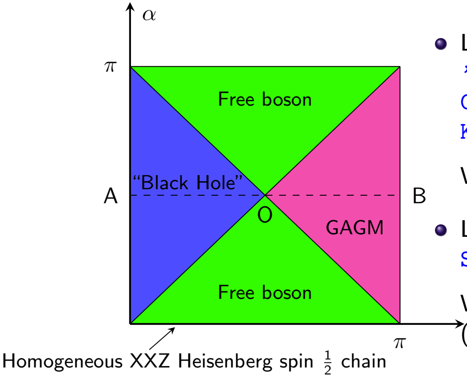
$$+\frac{\mathrm{i}}{2\cos(\gamma)}\,(\sigma^x_m\,\sigma^y_{m+2} - \sigma^y_m\sigma^x_{m+2})\,\sigma^z_{m+1} - \frac{\mathrm{i}}{2}\,(\sigma^x_m\,\sigma^y_{m+1} - \sigma^y_m\sigma^x_{m+1})\,(\sigma^z_{m+2} - \sigma^z_{m-1})\Big)\Bigg]$$

Phase diagram

$r = 2$ inhomogeneous XXZ spin $\frac{1}{2}$ chain with (quasi-)periodic BC

$$\eta_{2J} = \mathrm{e}^{\mathrm{i}\alpha}\,, \qquad \eta_{2J-1} = \mathrm{e}^{-\mathrm{i}\alpha} \quad \text{and} \quad q = \mathrm{e}^{\mathrm{i}\gamma}$$

with $\alpha, \gamma \in [0, \pi)$



- Line AO: [(Ikhlef), Jacobsen, Saluer '05; '06,'11; Frahm, Martins'12; Candu, Ikhlef'13; Bazhanov, GK, Koval, Lukyanov '19,'20]

  Whole BH region [Frahm, Seel'13]

- Line OB: [Ikhlef, Jacobsen, Saluer'09]

  Whole GAGM region [Kotousov, SL'21] (compact boson + 2 Majorana fermions)

Lattice Hamiltonians $\eta_1 = \eta_2^{-1} = \mathrm{e}^{\mathrm{i}\alpha}$, $q = \mathrm{e}^{\mathrm{i}\gamma}$ $(\frac{\pi}{2} < \gamma \leq \pi)$

$$\mathbb{H}^{(a)} = 2\mathrm{i}\zeta\,\partial_\zeta \log\big(\,\mathbb{T}(-q^{-1}\,\zeta/\eta_a)\,\big)\big|_{\zeta=1}$$

$$\mathbb{H} \equiv \mathbb{H}^{(1)} + \mathbb{H}^{(2)} \asymp N\,e_\infty + \frac{4\pi\,v_{\mathrm{F}}}{N}\,\big(\mathbf{I}_1^{(e)} + \bar{\mathbf{I}}_1^{(e)}\big) + o(N^{-1})$$

with

$$\mathbf{I}_1^{(e)} = \int \mathrm{d}u\,(\partial\varphi)^2 + \tfrac{\mathrm{i}}{2}\,\chi_1\partial\chi_1 + \tfrac{\mathrm{i}}{2}\,\chi_2\partial\chi_2$$

([Ikhlef, Jacobsen, Saluer'09] for $\alpha = \frac{\pi}{2}$)

What about scaling limit of $\mathbb{H}^{(1)} - \mathbb{H}^{(2)}$?

Lattice Hamiltonians [Kotousov, SL'21]

$$\mathbb{H}^{(a)} = 2\mathrm{i}\zeta\,\partial_\zeta \log\big(\,\mathbb{T}(-q^{-1}\,\zeta/\eta_a)\,\big)\Big|_{\zeta=1} \qquad\qquad (\eta_1 = \eta_2^{-1} = \mathrm{e}^{\mathrm{i}\alpha},\ q = \mathrm{e}^{\mathrm{i}\gamma})$$

$$\mathbb{H}^{(1)} + \mathbb{H}^{(2)} \asymp N\,e_\infty + \frac{4\pi v_{\mathrm{F}}}{N}\big(\mathbf{I}_1^{(e)} + \bar{\mathbf{I}}_1^{(e)}\big) + o(N^{-1})$$

$$\mathbb{H}^{(1)} - \mathbb{H}^{(2)} \asymp \frac{4\pi v_{\mathrm{F}}}{N}\Big(C_1\ \big(\mathbf{I}_1^{(e)} - \bar{\mathbf{I}}_1^{(e)}\big) + C_2\ \big(\mathbf{I}_1^{(o)} - \bar{\mathbf{I}}_1^{(o)}\big) + o(1)\Big)$$

with

$$\mathbf{I}_1^{(e)} = \int_0^{2\pi}\mathrm{d}u\,\Big((\partial\varphi)^2 + \tfrac{\mathrm{i}}{2}\,\chi_1\partial\chi_1 + \tfrac{\mathrm{i}}{2}\,\chi_2\partial\chi_2\Big)$$

$$\mathbf{I}_1^{(o)} = \int_0^{2\pi}\mathrm{d}u\,\Big(\rho\chi_2\,\partial\chi_1 - \mathrm{i}\tau\chi_2\chi_1\partial\varphi + \mathrm{i}\tau\,\chi_2\partial\chi_2\Big)$$

$C_1,\ C_2,\ \rho,\ \tau$ some constants depending on the lattice parameters (α, γ)

GAGM Hamiltonians

- $\widehat{\mathfrak{sl}}_1(2) \otimes \widehat{\mathfrak{sl}}_1(2)$ current algebra can be bosonized/fermionized in terms of $(\varphi, \chi_1, \chi_2)$. Then

$$\mathbf{I}_1^{(e)} = -\frac{1-\beta^2}{4\beta^2}\ \left(\mathbf{H}^{(1)}_{\rm gen} + \mathbf{H}^{(2)}_{\rm gen}\right)\ , \qquad \mathbf{I}_1^{(o)} = \frac{\rho}{4}\ \left(\mathbf{H}^{(1)}_{\rm gen} - \mathbf{H}^{(2)}_{\rm gen}\right)$$

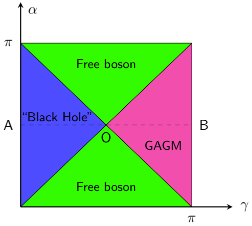
$$\mathbf{I}_1^{(e)} = \int_0^{2\pi} {\rm d}u \left((\partial\varphi)^2 + \tfrac{\rm i}{2}\,\chi_1\partial\chi_1 + \tfrac{\rm i}{2}\,\chi_2\partial\chi_2\right)$$

$$\mathbf{I}_1^{(o)} = \int_0^{2\pi} {\rm d}u \left(\rho\chi_2\,\partial\chi_1 - {\rm i}\tau\chi_2\chi_1\partial\varphi + {\rm i}\tau\,\chi_2\partial\chi_2\right)$$

- GAGM parameters (β, z_1, z_2) : $z_1 = z_2^{-1}$, $\quad x = \frac{1}{2}\,(z_1 + z_2)$.

$$\rho = \frac{1}{\sqrt{2}}\,(\beta^{-1} - \beta)\ , \quad \tau = -\frac{\rm i}{\sqrt{2}}\ (\beta^{-1} - \beta)\ \frac{x}{\sqrt{1-x^2}}$$

Lattice vs GAGM parameters



Lattice parameters (q, η_1, η_2) : $\quad q = \mathrm{e}^{\mathrm{i}\gamma},\ \eta_1 = \eta_2^{-1} = \mathrm{e}^{\mathrm{i}\alpha} \ \mapsto$

GAGM parameters (β, z_1, z_2) : $z_1 = z_2^{-1}\ , \qquad x = \frac{1}{2}\,(z_1 + z_2)$

$$\gamma = \frac{\pi}{2}\,(1+\beta^2) \qquad\qquad (0<\beta<1)$$

$$\alpha = \frac{\pi}{2} + \frac{1}{2\mathrm{i}(1+\xi)}\ \log\left[\frac{\mathrm{e}^{-\frac{\mathrm{i}\pi}{2}\xi}\,q(x) + \mathrm{e}^{+\frac{\mathrm{i}\pi}{2}\xi}\,q(-x)}{\mathrm{e}^{+\frac{\mathrm{i}\pi}{2}\xi}\,q(x) + \mathrm{e}^{-\frac{\mathrm{i}\pi}{2}\xi}\,q(-x)}\right] \qquad (0<x<1)$$

Here $\xi = \frac{\beta^2}{1-\beta^2}$ and

$$q(x) = \frac{\pi}{\sin(\pi\xi)}\ (1+x)\ {}_2F_1\big(1+\xi, -\xi, 2, \tfrac{1}{2}\,(1+x)\big)$$

Lattice Hamiltonians

$$\mathbb{H}^{(1)} + \mathbb{H}^{(2)} \asymp N\,e_\infty + \frac{4\pi v_{\rm F}}{N}\,\big(\mathbf{I}_1^{(e)} + \bar{\mathbf{I}}_1^{(e)}\big) + o(N^{-1})$$

$$\mathbb{H}^{(1)} - \mathbb{H}^{(2)} \asymp \frac{4\pi v_{\rm F}}{N}\Big(C_1\,\big(\mathbf{I}_1^{(e)} - \bar{\mathbf{I}}_1^{(e)}\big) + C_2\,\big(\mathbf{I}_1^{(o)} - \bar{\mathbf{I}}_1^{(o)}\big) + o(1)\Big)$$

$$C_1 = \frac{\mathrm{i}\,\xi(1+\xi)}{2\pi}\,f(x)\,f'(x)\,, \qquad C_2 = \frac{\xi(1+\xi)^2 f^2(x)}{2\pi\sqrt{1-x^2}}\,\sqrt{2}\beta$$

with

$$f(x) = \sqrt{q^2(x) + q^2(-x) + 2\cos(\pi\xi)\,q(x)\,q(-x)}$$

Here $\xi = \frac{\beta^2}{1-\beta^2}$ and

$$q(x) = \frac{\pi}{\sin(\pi\xi)}\,(1+x)\ {}_2F_1\big(1+\xi, -\xi, 2, \tfrac{1}{2}\,(1+x)\big)$$

Summary

- The Generalized Affine Gaudin Model,

$$\mathbf{H}^{(a)}_{\rm gen} = \int_0^{2\pi} \frac{\mathrm{d}u}{2\pi} \Bigg[\frac{\beta^2 K}{1-\beta^2}\ G^{(a)} + \frac{1}{4K}\ \frac{1-\beta}{1+\beta}\ \Big(k_a \big(J_0^{(\rm tot)}\big)^2 - K\, J_0^{(a)} J_0^{(\rm tot)} \Big)$$

$$-\sum_{\substack{b=1\\ b\neq a}}^{r} \frac{1}{z_a - z_b} \Big(\frac{1}{4}\,(z_a+z_b)\, J_0^{(a)} J_0^{(b)} + z_a\, J_+^{(b)} J_-^{(a)} + z_b\, J_+^{(a)} J_-^{(b)} - k_a z_b\, G^{(b)} - k_b z_a\, G^{(a)} \Big) \Bigg]$$

  governs the critical behavior of the inhomogeneous higher-spin six-vertex model. The corresponding local Boltzmann weights are contained in the R-matrix that is the trigonometric solution of the Yang-Baxter equation with the anisotropy parameter

$$q = -\,\mathrm{e}^{\frac{\mathrm{i}\pi}{K}(\beta^2-1)} \qquad \Big(K = \sum_{a=1}^{r} k_a\ , \quad k_a = 2\ell_a\Big)$$

  In the limit $\beta \to 1^-$ the trigonometric R-matrix becomes the rational one.

- The GAGM fits within the framework of the standard Yang-Baxter integrability. The Hamiltonians $\mathbf{H}^{(a)}_{\rm gen}$ are part of a large commuting family which, as usual, involves the quantum transfer-matrices and Baxter Q-operators.

Application: Fedya's approach to calculation of the correlation functions

CHAPTER 5

Applications and Generalisations

5.1. Function $\omega(\zeta, \xi|\alpha)$ via Integral Equation

The expression (4.35) for the function $\omega(\zeta, \xi|\alpha)$ which we gave in the previous chapter possesses many attractive features. However, it is not very convenient for considering the limit $\mathbf{n} \to \infty$ which is important for physical applications. In this section, we shall provide an alternative description for $\omega(\zeta, \xi|\alpha)$. We shall consider only the case where the *Matsubara* spins equal 1/2, but we shall keep inhomogeneities $\tau_{\mathbf{m}}$ which play an important role in considering various $\mathbf{n} \to \infty$ limits.

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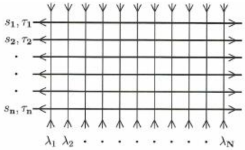

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Happy 65^{th}!