Solving the Form Factor Bootstrap for Solvable Irrelevant Deformations

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BASED ON WORK IN COLLABORATION WITH

Olalla Castro-Alvaredo **Fabio Sailis** **Isván M. Szécsényi**

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- Completing the Bootstrap Program for $\mathrm{T\overline{T}}$-Deformed Massive Integrable Quantum Field Theories
  ArXiv: 2305.17068

- Form Factors and Correlation Functions of $\mathrm{T\overline{T}}$-Deformed Integrable Quantum Field Theories
  JHEP 09 (2023) 048 | ArXiv: 2306.01640

- Entanglement Entropy from Form Factors in $\mathrm{T\overline{T}}$-Deformed Integrable Quantum Field Theories
  ArXiv: 2306.11064

- On the General Solution of Form Factor Equations
  To appear (hopefully) soon

(Generalised) $\mathrm{T}\overline{\mathrm{T}}$-deformations: robust extensions of proper local (Wilsonian) QFTs

Relations to 2D Quantum Gravity, (effective) String Theory, cutoff holography, etc…

See the excellent review by Y. Jiang [1904.13376]

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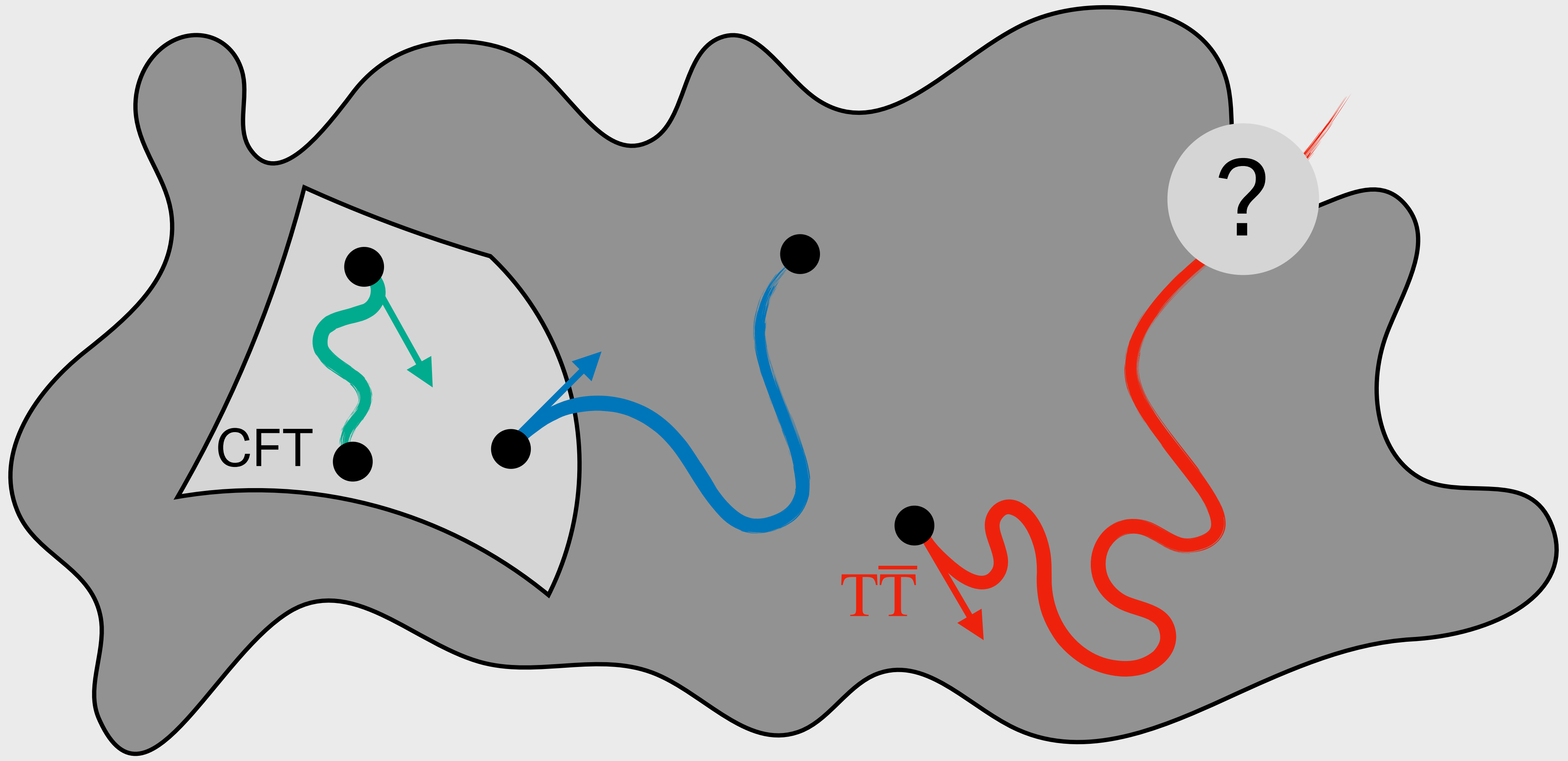
Personal angle

Geometric structure of the "theory space" (to be discovered)

Exploration of this space

Find new integrable systems

Here Be Dragons



A lot is known

Finite-size spectrum, partition functions, S-matrix, realisation as JT-gravity, ...

A lot is not known

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What about correlation functions? They are difficult to study (as usual)

Some results are out there (particularly on deformed CFTs)

Aharony, Vaknin [1803.00100] | Cardy [1907.03394] | Kruthoff, Parrikar [2006.03054] | He, Sun [2004.07486]

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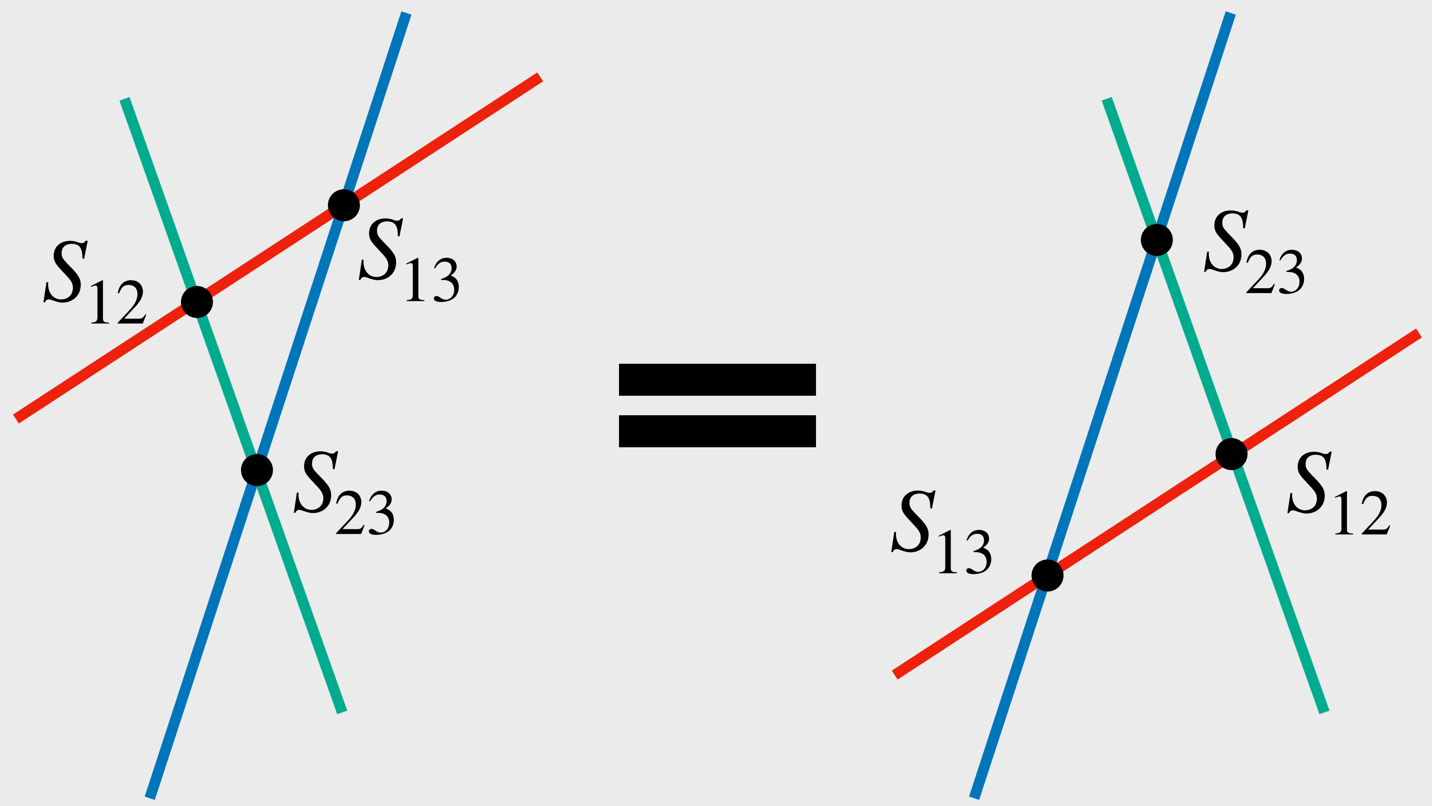
Important tool to understand the physics (non-locality, spectral decomposition, RG, …)

1] Computation of the exact (2 body) scattering matrix $S_{ab}(\theta)$



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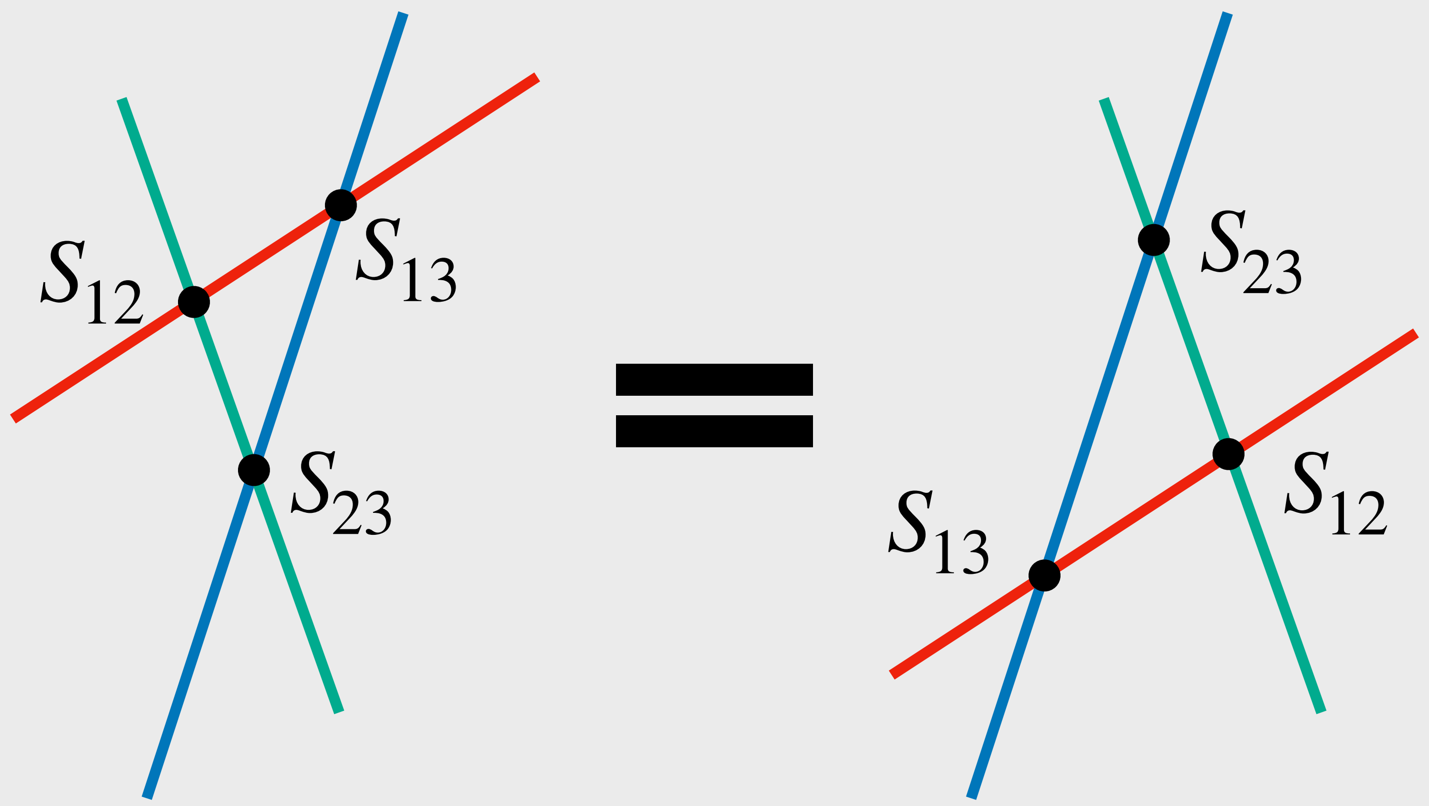
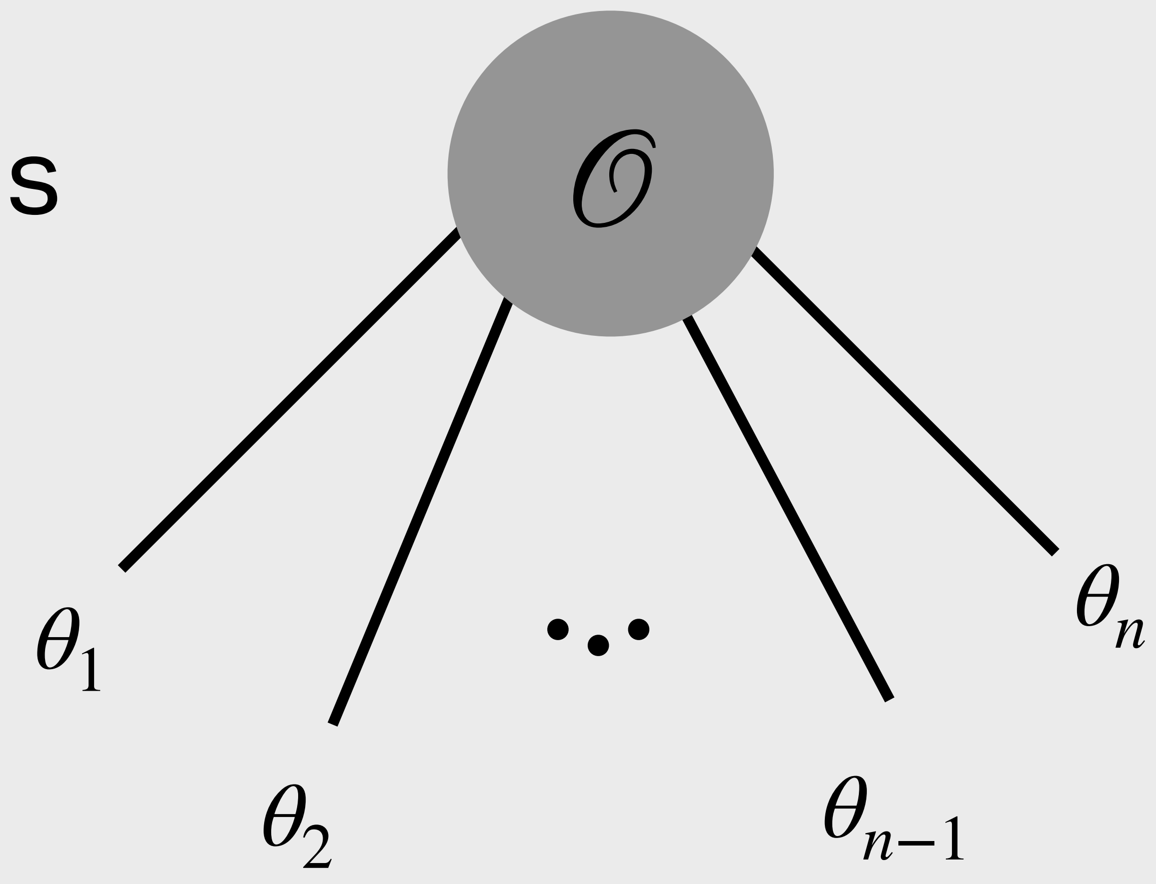


2] Computation of the Form Factors $F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n)$ of (semi-)local fields



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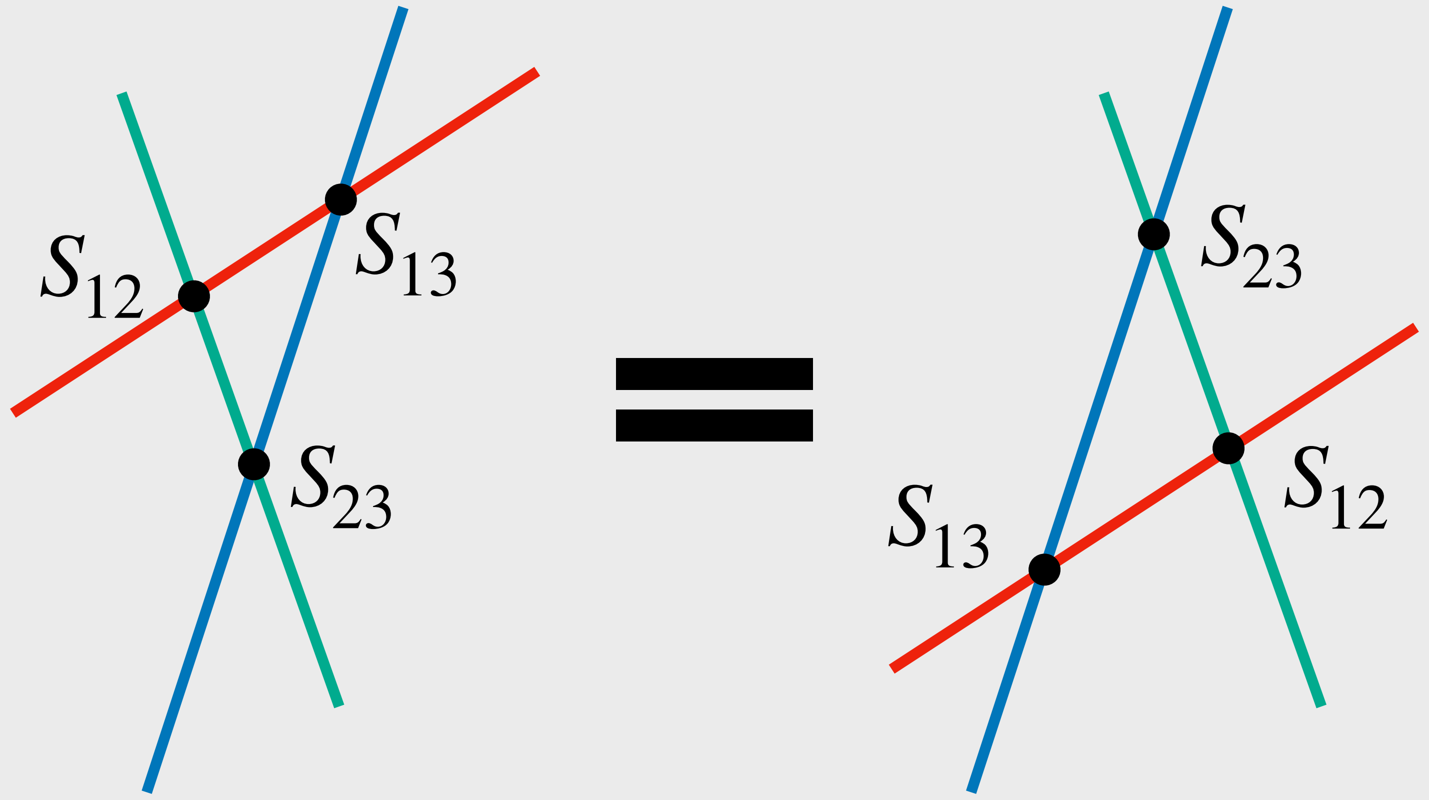

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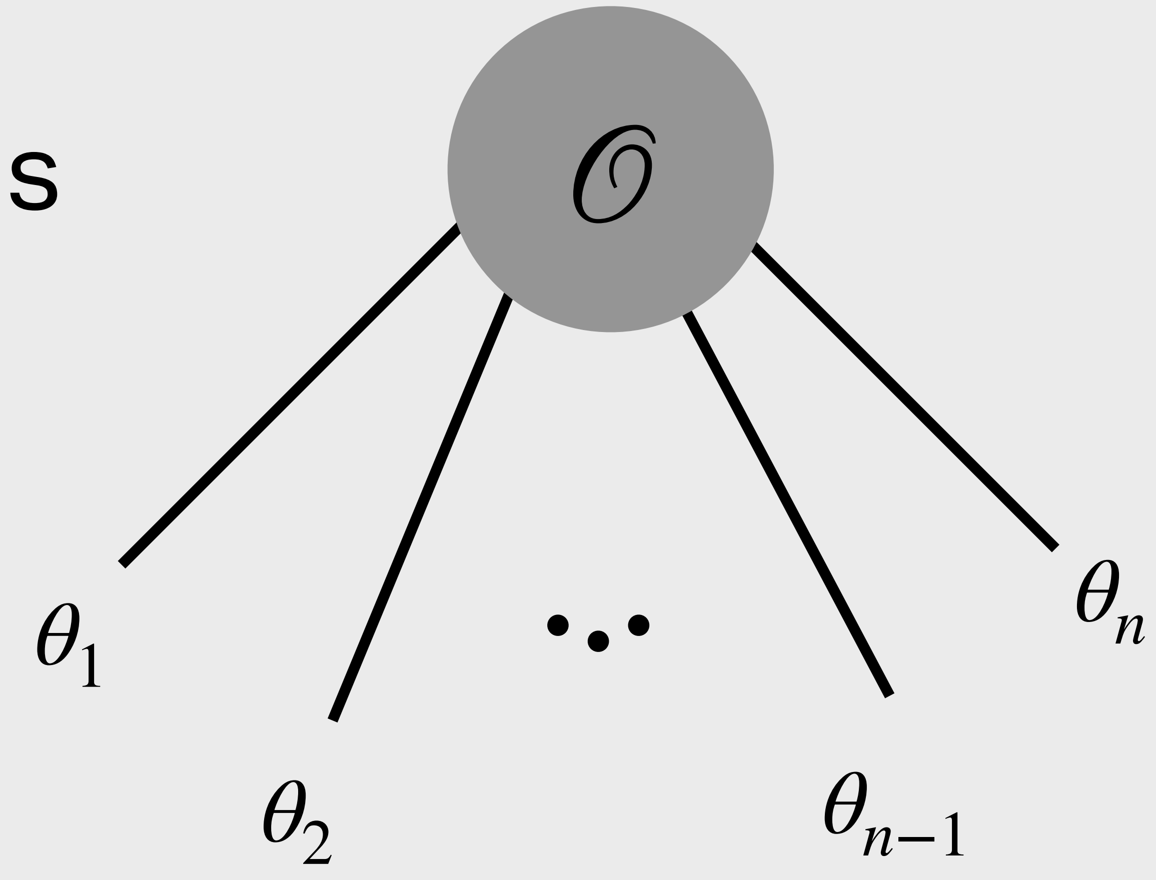

3] Computation of correlation functions and non-trivial checks

4] Classification of operators from solutions to FF equations

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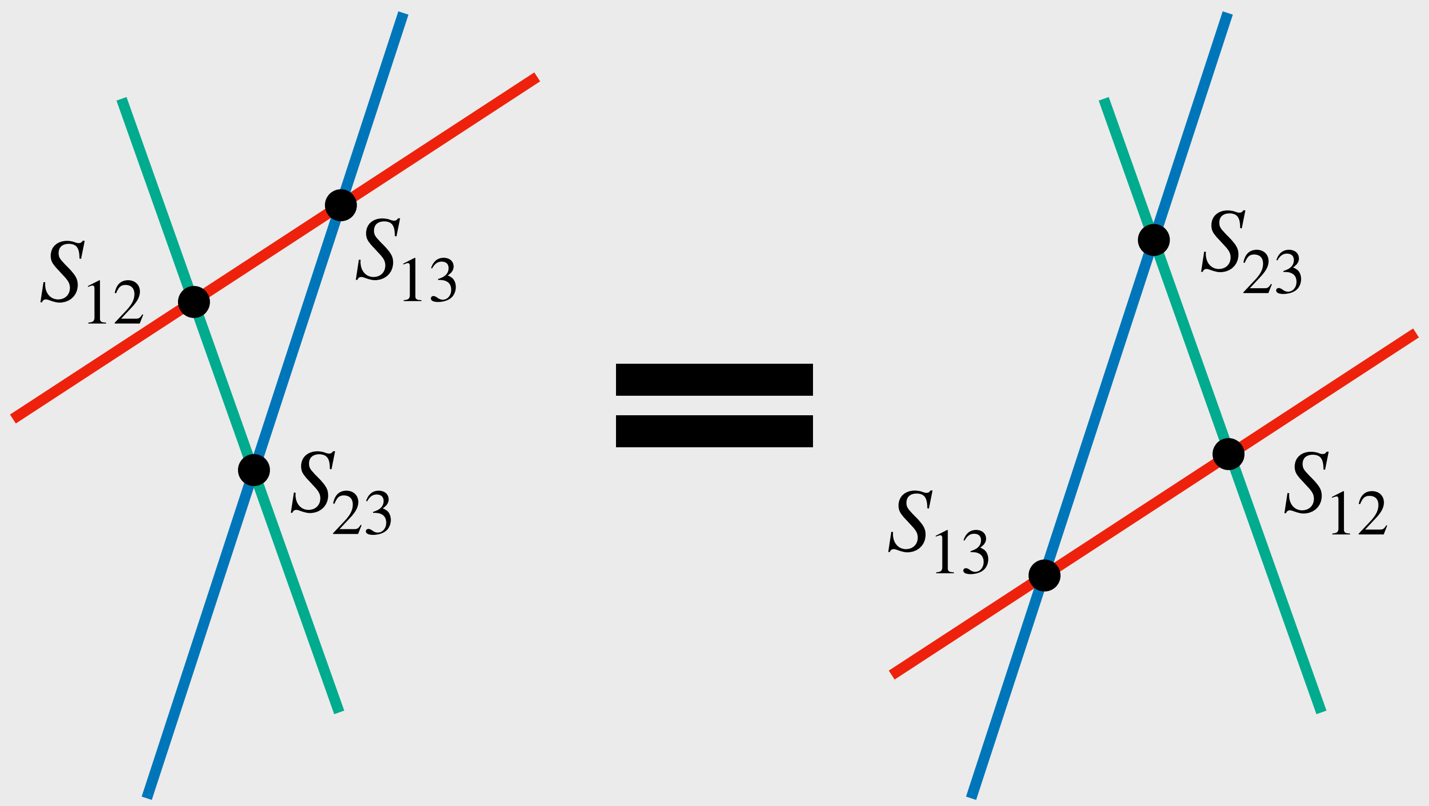

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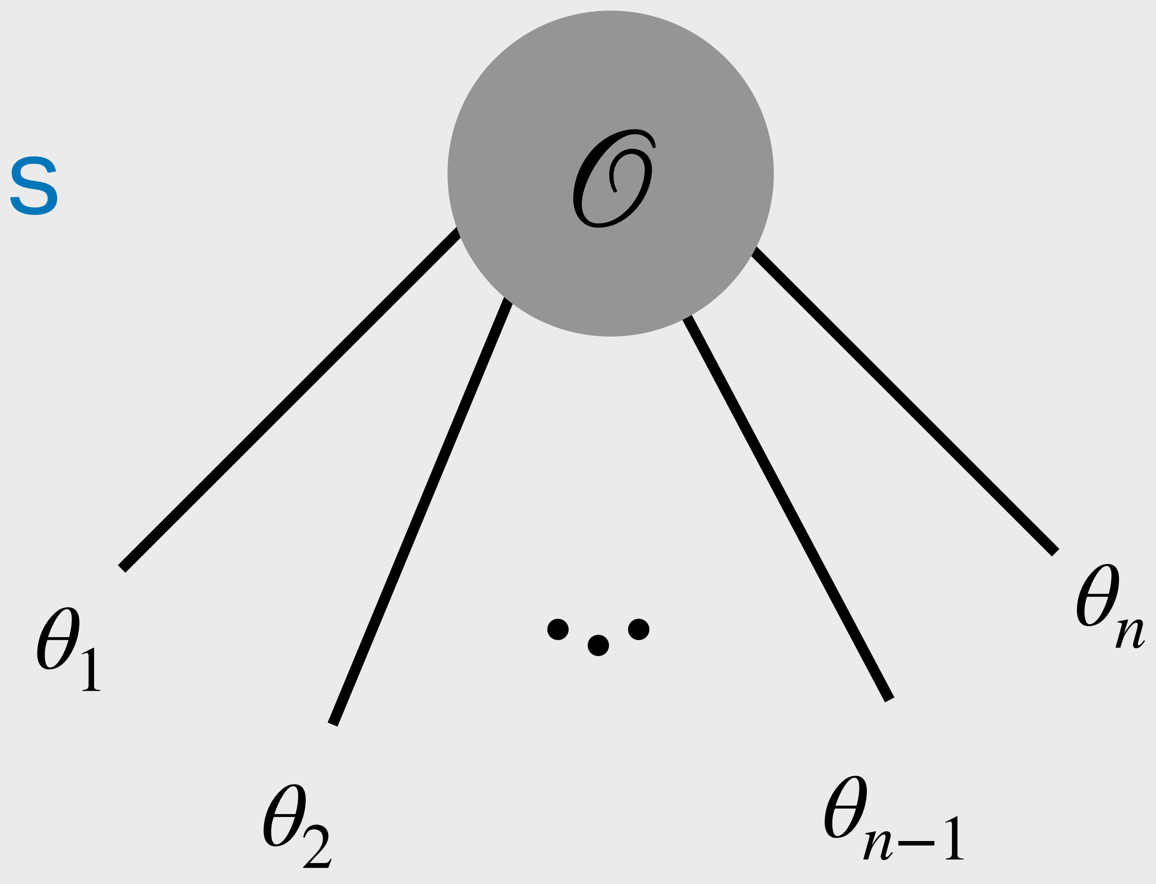

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4] Classification of operators from solutions to FF equations

Matrix elements of (semi-)local operators: $F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n) \doteq \left\langle 0 \middle| \mathcal{O}(0) \middle| \theta_1, \ldots, \theta_n \right\rangle$

THE FORM FACTOR EQUATIONS

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Focus on a theory with a single stable particle (no bound states)

$$F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_i,\theta_{i+1},\ldots,\theta_n) = S(\theta_{i+1}-\theta_i)F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_{i+1},\theta_i,\ldots,\theta_n)$$

$$F_n^{\mathcal{O}}(\theta_1+2\pi i,\theta_2\ldots,\theta_n) = \gamma_{\mathcal{O}}F_n^{\mathcal{O}}(\theta_2,\ldots,\theta_n,\theta_1)$$

$$\lim_{\vartheta\to\theta}(\theta-\vartheta)F_n^{\mathcal{O}}(\vartheta+\pi i,\theta,\theta_1,\ldots,\theta_n) = i\left(1-\gamma_{\mathcal{O}}\prod_{j=1}^{n}S(\theta-\theta_j)\right)F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_n)$$

For diagonal theories (no backscattering, e.g. Ising, Lee-Yang and sinh-Gordon)

Start with 2-particle “minimal" FF and construct the other from it

Karowski, Weiss ’78 | Smirnov ’92

Recipe: $S(\theta) = e^{-i\int_0^\infty \frac{dt}{t} g(t) \sin\left(\frac{\theta}{\pi}t\right)} \implies F_{\mathrm{min}}(\theta) = \mathcal{N} e^{\int_0^\infty \frac{dt}{t} \frac{g(t)}{\sinh t} \sin^2\left(\frac{i\pi - \theta}{2\pi}t\right)}$

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Example: sinh-Gordon

$$F_{\text{min,shG}}(\theta) = e^{-4\int_0^\infty \frac{dx}{x}\frac{\sinh\left(x\frac{1+b}{4}\right)\sinh\left(x\frac{1-b}{4}\right)\sinh\frac{x}{2}}{\sinh^2 x}\cos\left(x\frac{\vartheta}{\pi}\right)}$$

Fring, Mussardo, Simonetti ’93 | Koubek, Mussardo ’93 | Mussardo ’10

Family of theories generated by special flow equations

$$\frac{d}{d\alpha}\mathscr{A}_\alpha = \int d^2x\, X^{(\alpha)}_{\mathrm{AB}}(x)\,, \qquad X^{(\alpha)}_{\mathrm{AB}}(x) := \lim_{x'\to x} \varepsilon_{\mu\nu}\left[J^\mu_{\mathrm{A}}(x;\alpha)J^\nu_{\mathrm{B}}(x';\alpha) - J^\mu_{\mathrm{B}}(x;\alpha)J^\nu_{\mathrm{A}}(x';\alpha)\right]$$

$$\partial_\mu J^\mu_\bullet(x;\alpha) \simeq 0$$

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Cardy '19 | Dubovsky, Negro, Porrati '23

If J are internal symmetry currents $\Longrightarrow$ deformation is marginal

Space-time symmetries and higher conserved currents $\Longrightarrow$ deformation is irrelevant

Most famous example: the $\mathrm{T\overline{T}}$ deformation $\quad J^\mu_{\mathrm{A}}(x) = T^\mu_0(x)\,, \quad J^\mu_{\mathrm{B}}(x) = T^\mu_1(x)$

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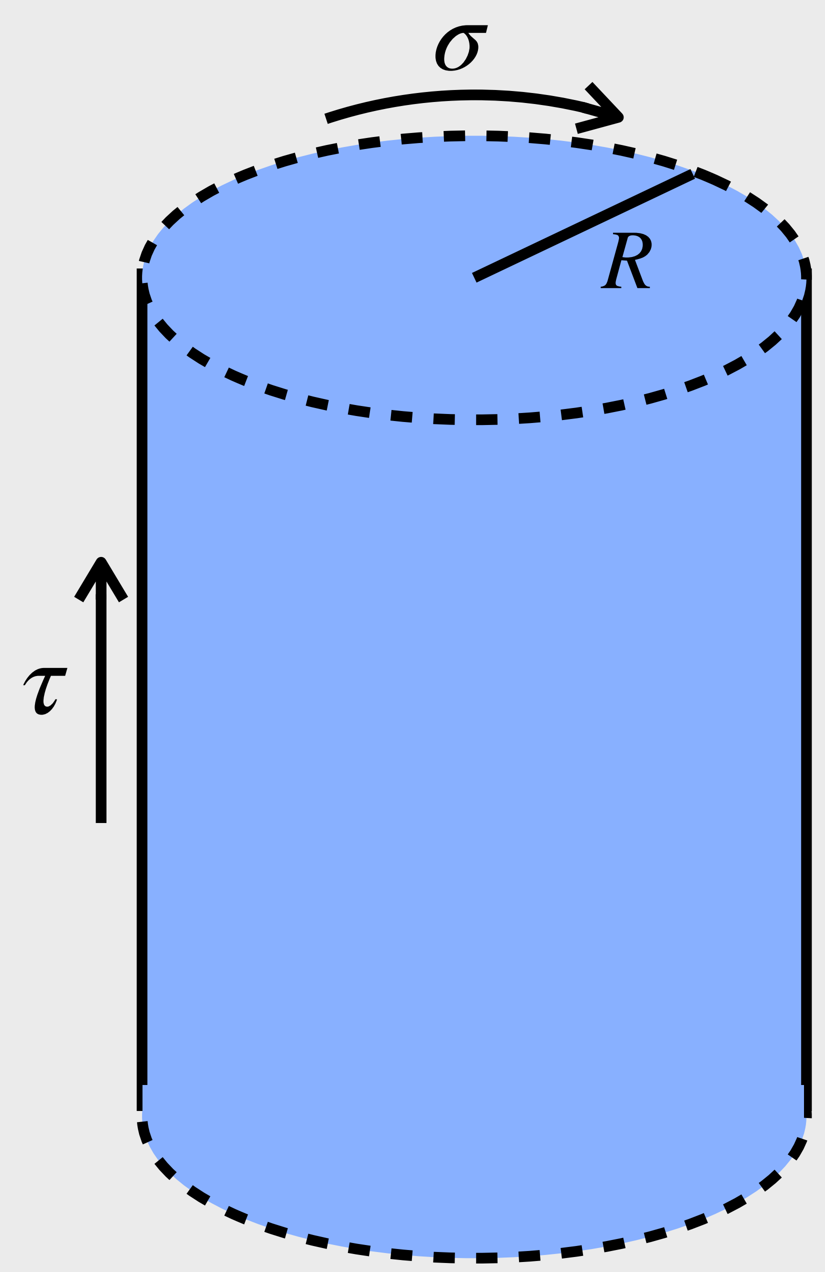
Striking example of solvability ($T\bar{T}$ case)

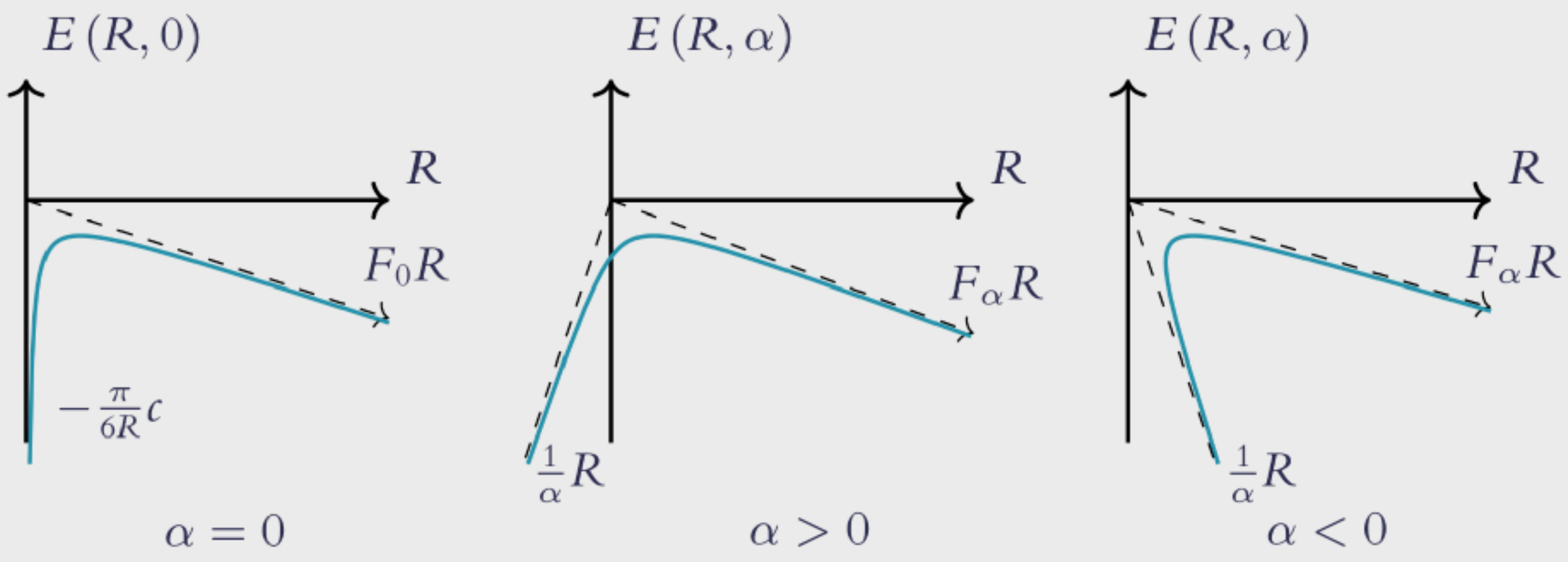
Smirnov, Zamolodchikov '16,
Cavaglià, Negro, Szécsényi, Tateo '16

Finite-size spectrum obeys the (inviscid, forced) Burgers equation

$$\frac{\partial}{\partial\alpha}E_n(R;\alpha) + E_n(R;\alpha)\frac{\partial}{\partial R}E_n(R;\alpha) + \frac{1}{R}P_n(R)^2 = 0$$



Resulting energy levels are not compatible with a UV fixed point



Alternative realisation for factorised scattering theories

Smirnov, Zamolodchikov '16

$$S_{\boldsymbol{\alpha}}(\theta) = S_0(\theta)\Phi_{\boldsymbol{\alpha}}(\theta)\ , \qquad \Phi_{\boldsymbol{\alpha}}(\theta) := \exp\left[-i\sum_{s\in\mathcal{S}}\alpha_s \sinh(s\theta)\right]$$

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$\mathcal{S}$ is the set of spins of local conserved charges (typically $\mathcal{S} \subseteq 2\mathbb{Z}_{\geq 0} + 1$)

Behaviour of Φ heavily depends on convergence properties of the series

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Assumption: $\alpha_s = 0$ for almost all $s \in \mathcal{S}$

S matrix has no nice integral representation!

Ansatz: the minimal form factor factorises

$$F_{\min}(\theta;\boldsymbol{\alpha}) = F_{\min}(\theta;\mathbf{0})\varphi_{\boldsymbol{\alpha}}(\theta) \quad \Longrightarrow \quad \varphi_{\boldsymbol{\alpha}}(\theta) = \Phi_{\boldsymbol{\alpha}}(\theta)\varphi_{\boldsymbol{\alpha}}(-\theta) = \varphi_{\boldsymbol{\alpha}}(2\pi i - \theta)$$

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Dubovsky, Flauger, Gorbenko '12 (for TTbar)

$$\varphi_{\boldsymbol{\alpha}}(\theta) = \exp\left[-\frac{i\pi - \theta}{2\pi}\sum_{s\in\mathcal{S}}\alpha_s \sinh(s\theta) + \sum_{t\in\mathbb{Z}}\beta_t \cosh(t\theta)\right]$$

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A physicist's pragmatic approach: just forget about them (a.k.a. "minimality")

What about higher particle FF?

They factorise as well!

$$F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_n;\boldsymbol{\alpha}) = F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_n;\mathbf{0})G_n^{\mathcal{O}}(\theta_1,\ldots,\theta_n;\boldsymbol{\alpha})$$

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Universal properties of G_n:

- Valid for any field with $\gamma_{\mathcal{O}} = \pm 1$ (e.g. local fields, symmetry fields)
- It further factorises in the product of an oscillatory function of the S matrix and of $\gamma_{\mathcal{O}}$ and a product of functions φ

Example: Thermal Ising (free Majorana fermion) $(F_{\rm min}(\theta;\mathbf{0}) = -\,i\sinh\theta/2)$

$$F^{\mu}_{2n}(\theta_1,\ldots,\theta_{2n};\boldsymbol{\alpha}) = i^n\langle\mu\rangle_{\boldsymbol{\alpha}}\sqrt{\prod_{i=1}^{2n}\cos\left(\sum_{s\in\mathcal{S}}\frac{\alpha_s}{2}\sum_{j=1}^{2n}\sinh(s\theta_{ij})\right)}\prod_{i<j}\tanh\frac{\theta_{ij}}{2}\varphi(\theta_{ij};\boldsymbol{\alpha})$$

$$F^{\sigma}_{2n+1}(\theta_1,\ldots,\theta_{2n+1};\boldsymbol{\alpha}) = i^n F^{\sigma}_1(\boldsymbol{\alpha})\sqrt{\prod_{i=1}^{2n+1}\cos\left(\sum_{s\in\mathcal{S}}\frac{\alpha_s}{2}\sum_{j=1}^{2n+1}\sinh(s\theta_{ij})\right)}\prod_{i<j}\tanh\frac{\theta_{ij}}{2}\varphi(\theta_{ij};\boldsymbol{\alpha})$$

The field Θ (trace of EM tensor) requires an additional complicated normalisation

FF are "building blocks" for correlation functions

A nice expression: the "cumulant expansion" (for fields with $\langle \mathcal{O} \rangle \neq 0$)

Smirnov '92

$$\log \frac{\langle \mathcal{O}(0)\mathcal{O}(r)\rangle_{\boldsymbol{\alpha}}}{\langle \mathcal{O}\rangle_{\boldsymbol{\alpha}}^2} \approx \int_{-\infty}^{\infty} d\theta\, K_0(2mr\cosh\frac{\theta}{2}) \left| F_2^{\mathcal{O}}(\theta;\boldsymbol{\alpha}) \right|^2 + \cdots$$

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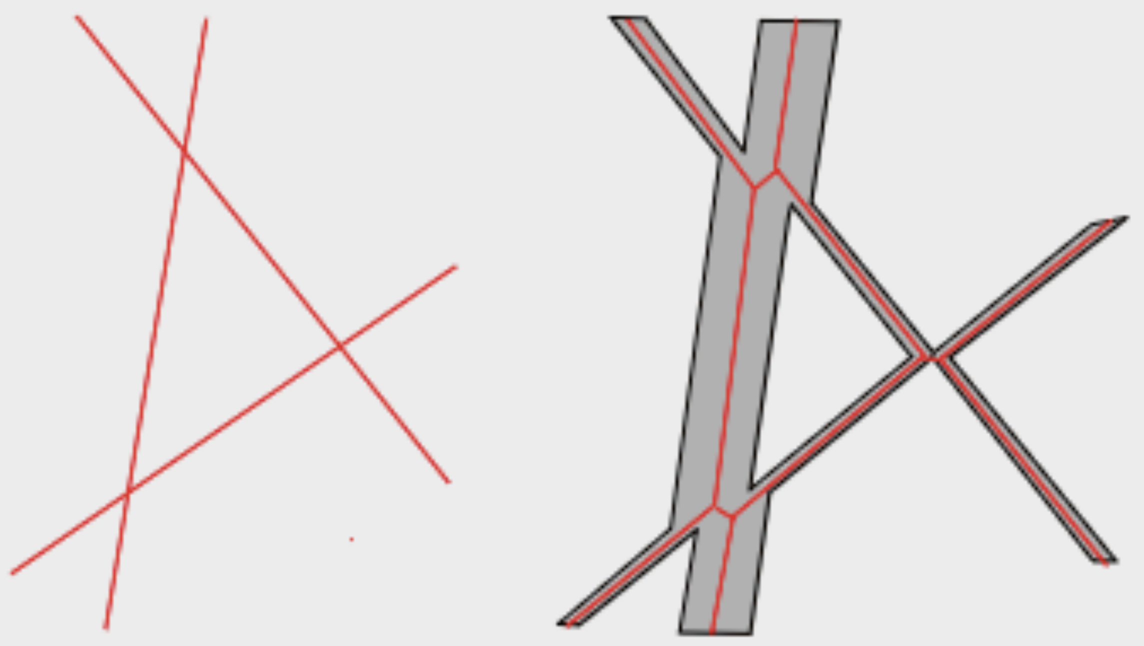
The presence of $\left|\varphi_{\boldsymbol{\alpha}}(\theta)\right|^2$ implies a behaviour $\propto e^{\frac{\theta}{\pi}\sum_s \alpha_s \sinh(s\theta)}$

Consequence:

$\alpha^\star > 0$: wild divergence

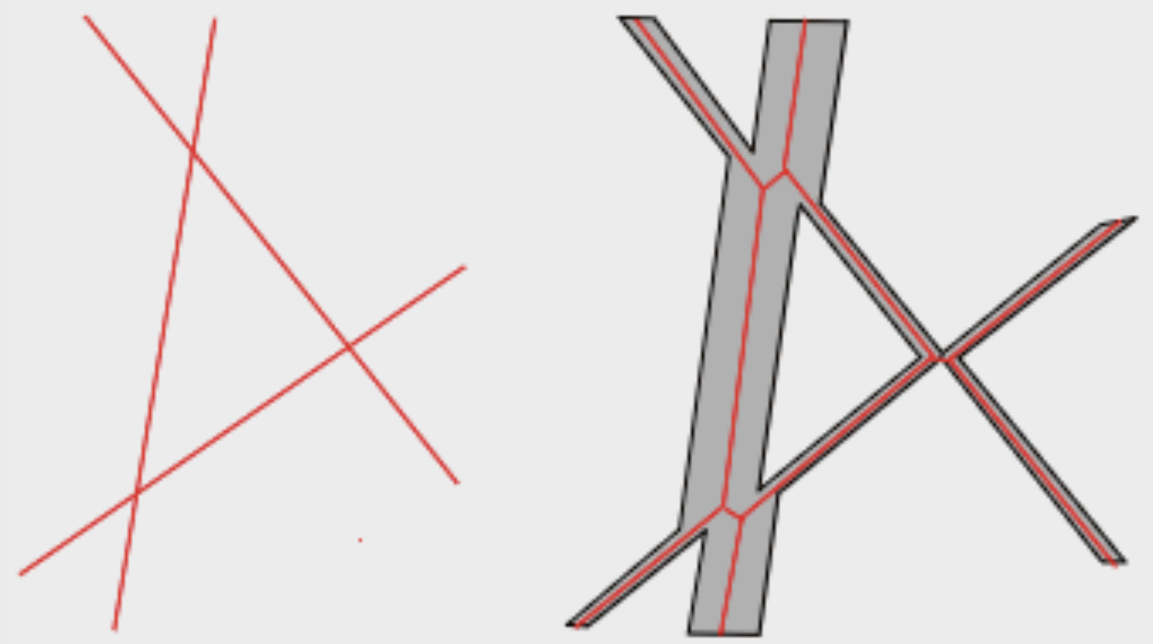
$\alpha^\star < 0$: hyper-convergence

Interpretation



Fundamental excitations acquire an effective size

Cardy, Doyon [2010.15733]



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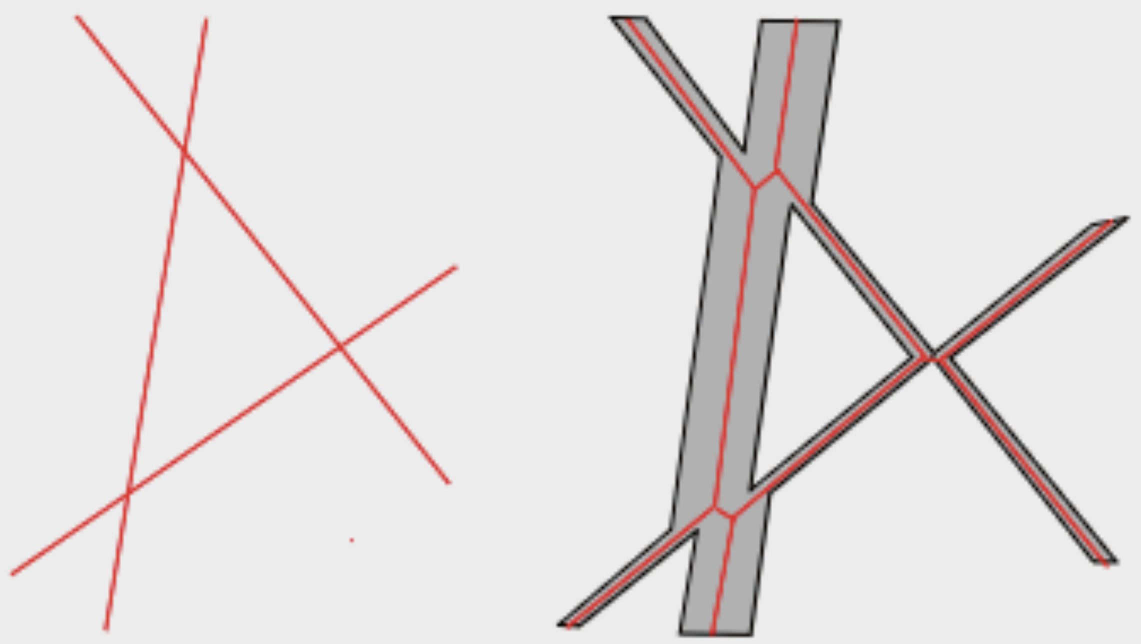
$\alpha^{\star} > 0$: effective size is **positive** $\Longrightarrow$ new scale limiting access to UV

High momentum (rapidity) particles produce divergences

Cure by introducing a cut-off $\Lambda = 2W_0(\pi r/\alpha)$ (Lambert function)



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$\alpha^\star < 0$: effective size is **negative** $\Longrightarrow$ UV can be probed without issue

Actually "there is more space"

Intuitively explains the hyper-convergence (and Hagedorn)

Consider the $T\bar{T}$ case for $\alpha < 0$

$$\log \frac{\langle \mathcal{O}(0)\mathcal{O}(r)\rangle_\alpha}{\langle \mathcal{O}\rangle_\alpha^2} \approx \int_{-\infty}^{\infty} d\theta\, K_0(2mr\cosh\frac{\theta}{2}) \left| F_2^{\mathcal{O}}(\theta;\alpha) \right|^2 + \cdots$$

Expand the Bessel function for $mr \ll 1$

$$\log \frac{\langle \mathcal{O}(0)\mathcal{O}(r)\rangle_\alpha}{\langle \mathcal{O}\rangle_\alpha^2} \approx -\log(mr) \int_{-\infty}^{\infty} d\theta\, f(\theta;\alpha) e^{\frac{\theta}{\pi}\alpha \sinh\theta} + \cdots = -4\Delta^{\mathcal{O}}(\alpha)\log(mr) + \cdots$$

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The 2-point functions appear to exhibit power-law scaling at small scales!

Tension with the expectations: there should be no conventional CFT in the UV

Consider the $\mathrm{T}\overline{\mathrm{T}}$ case for $\alpha < 0$

Consistency check: Zamolodchikov's c-theorem

Zamolodchikov '86

$$c^{\mathrm{UV}} - c^{\mathrm{IR}} = \frac{3}{2}\int_0^\infty dr\, r^3 \langle \Theta(0)\Theta(r)\rangle_{c,\alpha}$$

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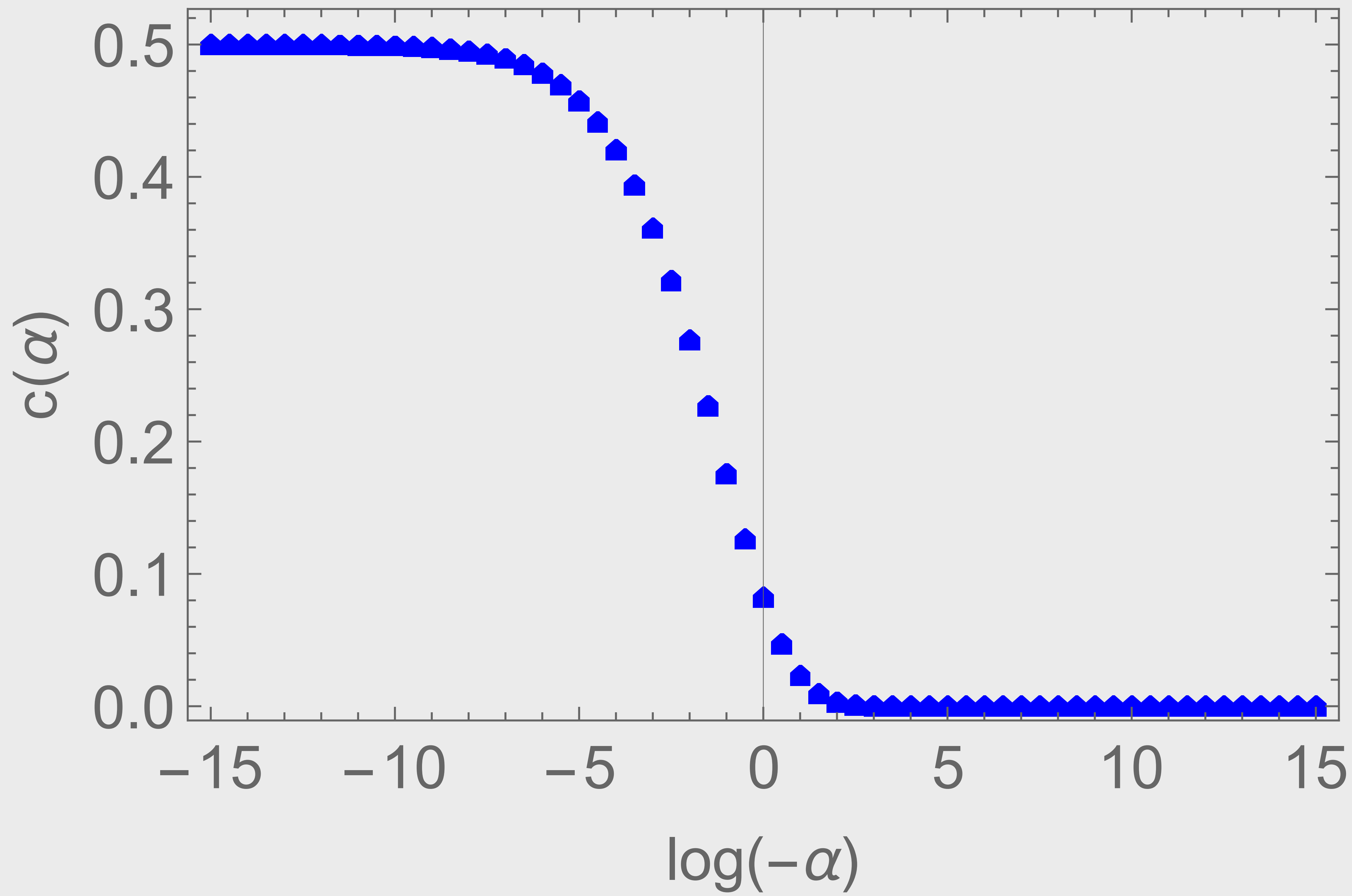
Insert our results in the Ising model case:

$$c(\alpha) = \frac{3}{8}\int_{-\infty}^{+\infty} dx \frac{\sin^2\left(\frac{\alpha}{2}\sinh x\right)}{\alpha^2 \cosh^6 \frac{x}{2}} e^{\frac{\alpha}{\pi} x \sinh x}$$



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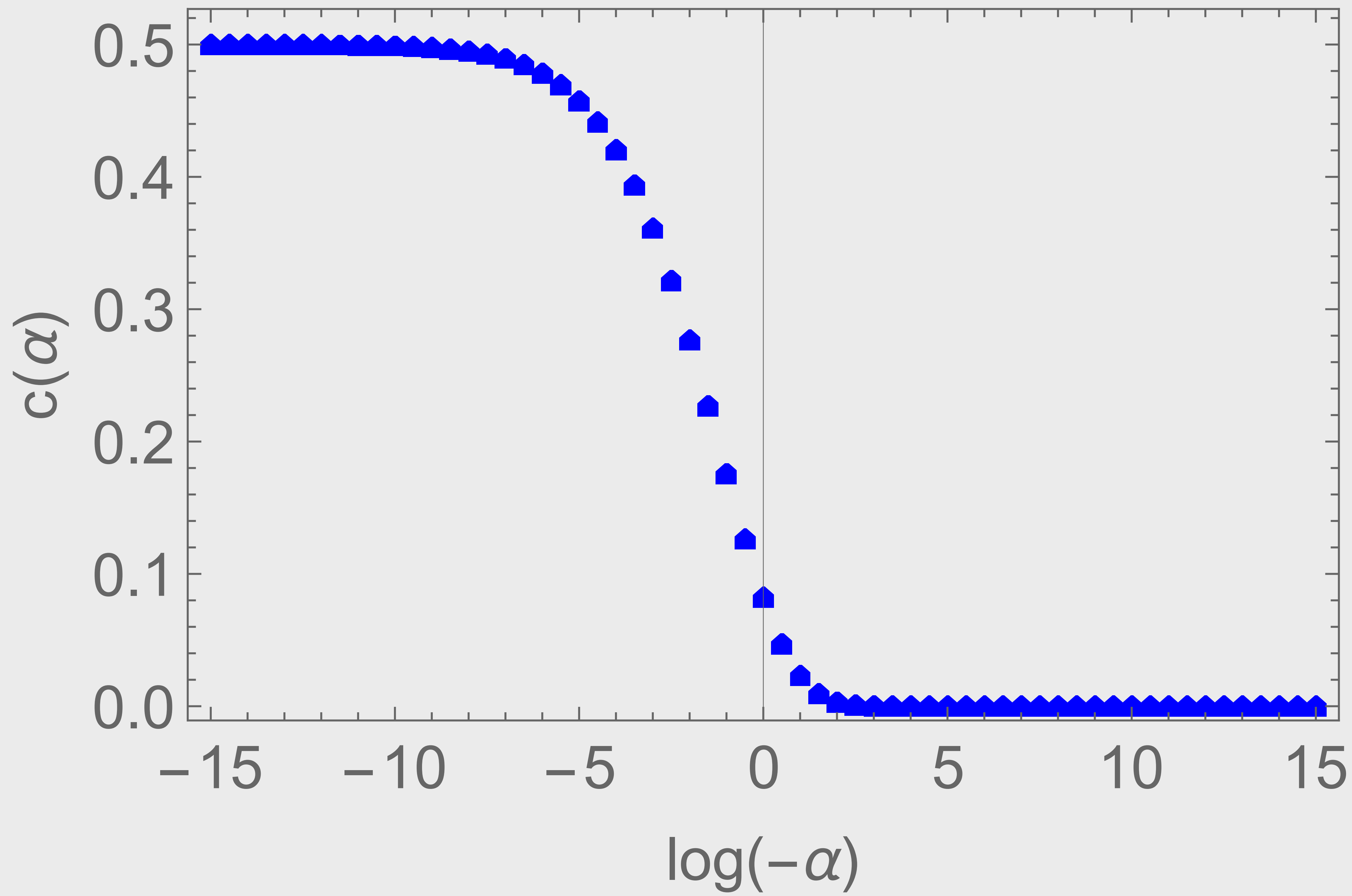

TBA tells us that $c^{\mathrm{UV}}(\alpha)$ should vanish for all $\alpha < 0$! Can we even define this quantity?

Throwing away all $\beta_t \cosh(t\theta)$ in $\varphi(\theta)$ is a strong assumption!

We have an example where they play a fundamental role: sinh-Gordon

WHAT ABOUT THE COSHES?

Throwing away all $\beta_t \cosh(t\theta)$ in $\varphi(\theta)$ is a strong assumption!

We have an example where they play a fundamental role: sinh-Gordon

$$S_{\text{shG}}(\theta) = \frac{\sinh\theta - i\cos\left(\frac{\pi}{2}b\right)}{\sinh\theta + i\cos\left(\frac{\pi}{2}b\right)} = -\exp\left[-4i\sum_{k=0}^{\infty}(-1)^{k+1}\frac{\cos\left(\frac{2k+1}{2}\pi b\right)}{2k+1}\sinh\big((2k+1)\theta\big)\right]$$

Fine-tuned superposition of solvable irrelevant deformations of thermal Ising

LeClair [2107.02230] | Ahn, LeClair [2205.10905]

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$$C_{\text{shG}}(\theta) = \frac{\log 2}{2} - \frac{1+b}{2}\log\left[\sin\frac{\pi b}{2} - \cosh\theta\right] - \frac{1-b}{2}\log\left[-\sin\frac{\pi b}{2} - \cosh\theta\right] +$$

$$-\frac{i}{4\pi}\left[\left(\text{Li}_2\left(ie^{-\theta-i\frac{\pi b}{2}}\right) + \text{Li}_2\left(-ie^{-\theta-i\frac{\pi b}{2}}\right) + (b\to -b)\right) + (\theta\to-\theta)\right]$$

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$$\sum_t \beta_t \cosh(t\theta)$$

In this case the β_t are related to the α_s

$$S(\theta) = e^{-i\int_0^\infty \frac{dt}{t} g(t) \sin\left(\frac{\theta}{\pi} t\right)} \quad \text{and} \quad g(t) = -t^2 \sum_{n=1}^{\infty} \frac{g_n}{t^2 + n^2\pi^2} \quad \Longrightarrow \quad \boxed{\alpha_s = -\frac{1}{2\pi}\frac{g_s}{s^2}}$$

$$F_{\min}(\theta) = e^{\int_0^\infty \frac{dt}{t} \frac{g(t)}{\sinh t} \cos\left(\frac{i\pi - \theta}{\pi} t\right)} \quad \Longrightarrow \quad \beta_t = \frac{1}{2\pi}\frac{\alpha_t}{t} - \frac{2}{\pi} t \sum_{s=1, s\neq t}^{\infty} \frac{g_s}{s^2 - t^2}$$

Can we take this as a definition?

$$S_{\boldsymbol{\alpha}}(\theta) = \exp\left[-i\sum_{s\in\mathcal{S}}\alpha_s\sinh(s\theta)\right] \Longrightarrow F_{\min}(\theta) = \exp\left[-\frac{i\pi-\theta}{2\pi}\sum_s\alpha_s\sinh(s\theta) + \sum_t\beta_t\cosh(t\theta)\right]$$

With $$\beta_t = \frac{1}{2\pi}\frac{\alpha_t}{t} - \frac{2}{\pi}t\sum_{s=1,s\neq t}^{\infty}\frac{g_s}{s^2-t^2}$$

For $\mathrm{T\overline{T}}$-deformed Ising $\Longrightarrow$ $$F_{\min}(\theta) = -\frac{\alpha}{2\pi}\left[\cosh\theta\log\left(2\cosh\theta-2\right)+1-\theta\sinh\theta+i\pi e^{\theta}\right]$$

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Extension to “twist fields” and computation of entanglement measures: straightforward

Castro-Alvaredo, Negro, Sailis [2306.11064] | Hou, He, Jiang [2306.07784]

Extension to theories w/ bound states and/or non-diagonal scattering

Thank you

Happy Birthday Fedor!