Solving the Form Factor Bootstrap for Solvable Irrelevant Deformations
SOLVING THE FORM FACTOR BOOTSTRAP FOR SOLVABLE IRRELEVANT DEFORMATIONS

BASED ON WORK IN COLLABORATION WITH

Olalla Castro-Alvaredo  Fabio Sailis  Isván M. Szécsényi
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BASED ON WORK IN COLLABORATION WITH

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✧ Completing the Bootstrap Program for $T\bar{T}$-Deformed Massive Integrable Quantum Field Theories
  ArXiv: 2305.17068

✧ Form Factors and Correlation Functions of $T\bar{T}$-Deformed Integrable Quantum Field Theories
  JHEP 09 (2023) 048  |  ArXiv: 2306.01640

✧ Entanglement Entropy from Form Factors in $T\bar{T}$-Deformed Integrable Quantum Field Theories
  ArXiv: 2306.11064

✧ On the General Solution of Form Factor Equations
  To appear (hopefully) soon
SOLVING THE FORM FACTOR BOOTSTRAP FOR SOLVABLE IRRELEVANT DEFORMATIONS

MOTIVATIONS

(Generalised) $\mathcal{T}\mathcal{T}$-deformations: robust extensions of proper local (Wilsonian) QFTs

Relations to 2D Quantum Gravity, (effective) String Theory, cutoff holography, etc...

See the excellent review by Y. Jiang [1904.13376]
(Generalised) $T\bar{T}$-deformations: robust extensions of proper local (Wilsonian) QFTs

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Personal angle

Geometric structure of the “theory space” (to be discovered)

Exploration of this space

Find new integrable systems
A lot is known

Finite-size spectrum, partition functions, $S$-matrix, realisation as JT-gravity, ...
A lot is not known

Finite-size spectrum, partition functions, S-matrix, realisation as JT-gravity, …

What about correlation functions? They are difficult to study (as usual)

Some results are out there (particularly on deformed CFTs)


Some results do not agree

E.g. Cardy [1907.03394] vs. Aharony, Barel [2304.14091]
A lot is not known

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Important tool to understand the physics (non-locality, spectral decomposition, RG, …)
1] Computation of the exact (2 body) scattering matrix $S_{ab}(\theta)$
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2] Computation of the Form Factors $F_n^\mathcal{O}(\theta_1, \ldots, \theta_n)$ of (semi-)local fields
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2] Computation of the Form Factors $F_n^\theta(\theta_1, \ldots, \theta_n)$ of (semi-)local fields

3] Computation of correlation functions and non-trivial checks

4] Classification of operators from solutions to FF equations
1] Computation of the exact (2 body) scattering matrix $S_{ab}(\theta)$

2] Computation of the Form Factors $F_n^\mathcal{O}(\theta_1, \ldots, \theta_n)$ of (semi-)local fields

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4] Classification of operators from solutions to FF equations
Matrix elements of (semi-)local operators: \[ F_n^\mathcal{O}(\theta_1, \ldots, \theta_n) \doteq \langle 0 \left| \mathcal{O}(0) \right| \theta_1, \ldots, \theta_n \rangle \]
Matrix elements of (semi-)local operators: 

\[ F_n^\mathcal{O}(\theta_1, \ldots, \theta_n) \equiv \left\langle 0 \left| \mathcal{O}(0) \right| \theta_1, \ldots, \theta_n \right\rangle \]

Focus on a theory with a single stable particle (no bound states)

\[ F_n^\mathcal{O}(\theta_1, \ldots, \theta_i, \theta_{i+1}, \ldots, \theta_n) = S(\theta_{i+1} - \theta_i) F_n^\mathcal{O}(\theta_1, \ldots, \theta_{i+1}, \theta_i, \ldots, \theta_n) \]

\[ F_n^\mathcal{O}(\theta_1 + 2\pi i, \theta_2, \ldots, \theta_n) = \gamma_\mathcal{O} F_n^\mathcal{O}(\theta_2, \ldots, \theta_n, \theta_1) \]

\[ \lim_{\theta \to \theta} (\theta - \theta) F_n^\mathcal{O}(\theta + \pi i, \theta, \theta_1, \ldots, \theta_n) = i \left( 1 - \gamma_\mathcal{O} \prod_{j=1}^{n} S(\theta - \theta_j) \right) F_n^\mathcal{O}(\theta_1, \ldots, \theta_n) \]
For diagonal theories (no backscattering, e.g. Ising, Lee-Yang and sinh-Gordon)

Start with 2-particle “minimal” FF and construct the other from it

Karowski, Weiss ‘78 | Smirnov ‘92

Recipe:

\[ S(\theta) = e^{-i \int_0^\infty \frac{dt}{t} g(t) \sin \left( \frac{\theta}{\pi} t \right)} \implies F_{\text{min}}(\theta) = \mathcal{N} e^{\int_0^\infty \frac{dt}{t} g(t) \sinh t \sin^2 \left( \frac{i\pi - \theta}{2\pi} t \right)} \]
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Example: sinh-Gordon

\[ F_{\text{min,shG}}(\theta) = e^{-4 \int_0^{\infty} \frac{dx}{x} \frac{\sinh \left( \frac{x + b}{4} \right) \sinh \left( \frac{x - b}{4} \right) \sinh \frac{x}{2}}{\sinh^2 x} \cos \left( x \frac{\theta}{\pi} \right)} \]

Fring, Mussardo, Simonetti ’93 | Koubek, Mussardo ’93 | Mussardo ’10
Solving the Form Factor Bootstrap for Solvable Irrelevant Deformations

Solvable Irrelevant Deformations

Family of theories generated by special flow equations

\[
\frac{d}{d\alpha} \mathcal{A}_\alpha = \int d^2x \, X^{(\alpha)}_{AB}(x), \quad X^{(\alpha)}(x) := \lim_{x' \to x} \epsilon_{\mu\nu} \left[ J^\mu_A(x; \alpha) J^\nu_B(x'; \alpha) - J^\mu_B(x; \alpha) J^\nu_A(x'; \alpha) \right]
\]

\[
\partial_\mu J^\mu_0(x; \alpha) \approx 0
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Family of theories generated by special flow equations

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\]

\[\partial_\mu J^\mu_\star(x; \alpha) \simeq 0\]

If \( J \) are internal symmetry currents \( \implies \) deformation is marginal

Space-time symmetries and higher conserved currents \( \implies \) deformation is irrelevant

Most famous example: the \( T\bar{T} \) deformation \( J^\mu_A(x) = T^\mu_0(x), \quad J^\mu_B(x) = T^\mu_1(x) \)
Family of theories generated by special flow equations

\[ \frac{d}{d\alpha} A_{\alpha} = \int d^2 x \, X^{(\alpha)}_{AB}(x) , \quad X^{(\alpha)}_{AB}(x) := \lim_{x' \to x} \varepsilon_{\mu\nu} \left[ J_{A}^{\mu}(x; \alpha) J_{B}^{\nu}(x'; \alpha) - J_{B}^{\mu}(x; \alpha) J_{A}^{\nu}(x'; \alpha) \right] \]

\[ \partial_{\mu} J_{\mu}^{\alpha}(x; \alpha) \simeq 0 \]

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Most famous example: the \( T\bar{T} \) deformation

\[ J_{A}^{\mu}(x) = T_{0}^{\mu}(x) , \quad J_{B}^{\mu}(x) = T_{1}^{\mu}(x) \]
Striking example of solvability (TT case)

Finite-size spectrum obeys the (inviscid, forced) Burgers equation

\[ \frac{\partial}{\partial \alpha} E_n(R; \alpha) + E_n(R; \alpha) \frac{\partial}{\partial R} E_n(R; \alpha) + \frac{1}{R} P_n(R)^2 = 0 \]

Resulting energy levels are not compatible with a UV fixed point
Alternative realisation for factorised scattering theories

\[
S_\alpha(\theta) = S_0(\theta) \Phi_\alpha(\theta) , \quad \Phi_\alpha(\theta) := \exp \left[ -i \sum_{s \in \mathcal{S}} \alpha_s \sinh(s\theta) \right]
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Smirnov, Zamolodchikov ’16
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\(\mathcal{S}\) is the set of spins of local conserved charges (typically \(\mathcal{S} \subseteq 2\mathbb{Z}_{\geq 0} + 1\))

Behaviour of \(\Phi\) heavily depends on convergence properties of the series

Smirnov, Zamolodchikov ’16

Camilo, Fleury, Lencsés, Negro, Zamolodchikov, ’21
Alternative realisation for factorised scattering theories

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\( S \) is the set of spins of local conserved charges (typically \( S \subseteq 2 \mathbb{Z}_{\geq 0} + 1 \))

Behaviour of \( \Phi \) heavily depends on convergence properties of the series

Assumption: \( \alpha_s = 0 \) for almost all \( s \in S \)

Smirnov, Zamolodchikov ’16

Camilo, Fleury, Lencsés, Negro, Zamolodchikov, ’21
S matrix has no nice integral representation!

Ansatz: the minimal form factor factorises

\[ F_{\min}(\theta; \alpha) = F_{\min}(\theta; 0) \varphi_{\alpha}(\theta) \implies \varphi_{\alpha}(\theta) = \Phi_{\alpha}(\theta) \varphi_{\alpha}(-\theta) = \varphi_{\alpha}(2\pi i - \theta) \]
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Most general solution is

\[
\varphi_{\alpha}(\theta) = \exp\left[ -\frac{i\pi - \theta}{2\pi} \sum_{s \in \mathcal{S}} \alpha_s \sinh(s \theta) + \sum_{t \in \mathbb{Z}} \beta_t \cosh(t \theta) \right]
\]

Dubovsky, Flauger, Gorbenko ‘12 (for TTbar)
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Universal
S matrix has no nice integral representation!

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\]

Universal \quad \text{Unusual } e^{\theta e^{s\theta}} \text{ dependence}

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$$

Universal $e^{\theta e^{\vartheta \theta}}$ dependence

Unusual $e^{\theta e^{s\theta}}$ dependence

Huge indeterminacy
SOLVING THE FORM FACTOR BOOTSTRAP FOR SOLVABLE IRRELEVANT DEFORMATIONS

FORM FACTORS FOR SOLVABLE IRRELEVANT DEFORMATIONS

\( S \) matrix has no nice integral representation!

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\]

Universal \hspace{1cm} Unusual \( e^{\theta e^{i\theta}} \) dependence \hspace{1cm} Huge indeterminacy

A physicist’s pragmatic approach: just forget about them (a.k.a. “minimality”)
What about higher particle FF?

They factorise as well!

\[
F_n^\theta(\theta_1, \ldots, \theta_n; \alpha) = F_n^\theta(\theta_1, \ldots, \theta_n; 0)G_n^\theta(\theta_1, \ldots, \theta_n; \alpha)
\]
What about higher particle FF?

They factorise as well!

\[ F_n^\circ(\theta_1, \ldots, \theta_n; \alpha) = F_n^\circ(\theta_1, \ldots, \theta_n; 0)G_n^\circ(\theta_1, \ldots, \theta_n; \alpha) \]

Universal properties of \( G_n \):

✧ Valid for any field with \( \gamma^\circ = \pm 1 \) (e.g. local fields, symmetry fields)

✧ It further factorises in the product of an oscillatory function of the \( S \) matrix and of \( \gamma^\circ \) and a product of functions \( \varphi \)
Example: Thermal Ising (free Majorana fermion) \( (F_{\text{min}}(\theta; \theta) = -i \sinh \theta/2) \)

\[
F^\mu_{2n}(\theta_1, \ldots, \theta_{2n}; \alpha) = i^n \langle \mu \rangle_\alpha \sqrt{\prod_{i=1}^{2n} \cos \left( \sum_{s \in \mathcal{S}} \frac{\alpha_s}{2} \sum_{j=1}^{2n} \sinh(s \theta_{ij}) \right) \prod_{i<j} \tanh \frac{\theta_{ij}}{2} \varphi(\theta_{ij}; \alpha)}
\]

\[
F^\sigma_{2n+1}(\theta_1, \ldots, \theta_{2n+1}; \alpha) = i^n F^\sigma_1(\alpha) \sqrt{\prod_{i=1}^{2n+1} \cos \left( \sum_{s \in \mathcal{S}} \frac{\alpha_s}{2} \sum_{j=1}^{2n+1} \sinh(s \theta_{ij}) \right) \prod_{i<j} \tanh \frac{\theta_{ij}}{2} \varphi(\theta_{ij}; \alpha)}
\]

The field \( \Theta \) (trace of EM tensor) requires an additional complicated normalisation.
FF are “building blocks” for correlation functions

A nice expression: the “cumulant expansion” (for fields with $\langle \mathcal{O} \rangle \neq 0$)

$$\log \frac{\langle \mathcal{O}(0) \mathcal{O}(r) \rangle_\alpha}{\langle \mathcal{O} \rangle^2_\alpha} \approx \int_{-\infty}^{\infty} d\theta \ K_0(2mr \cosh \frac{\theta}{2}) \left| F^\mathcal{O}_2(\theta; \alpha) \right|^2 + \cdots$$

Smirnov ‘92
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Smirnov '92

The presence of $\left| \varphi_\alpha(\theta) \right|^2$ implies a behaviour $\propto e^{\frac{\theta}{\pi} \sum_s \alpha_s \sinh(s\theta)}$

Consequence:

$\alpha^* > 0$: wild divergence

$\alpha^* < 0$: hyper-convergence
SOLVING THE FORM FACTOR BOOTSTRAP FOR SOLVABLE IRRELEVANT DEFORMATIONS
CORRELATION FUNCTIONS

Interpretation

Fundamental excitations acquire an effective size
Cardy, Doyon [2010.15733]
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Fundamental excitations acquire an effective size

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$\alpha^* > 0$: effective size is **positive** $\implies$ new scale limiting access to UV

High momentum (rapidity) particles produce divergences

Cure by introducing a cut-off $\Lambda = 2W_0(\pi r/\alpha)$ (Lambert function)
Interpretation

Fundamental excitations acquire an effective size
Cardy, Doyon [2010.15733]

\[ \alpha^* > 0: \text{effective size is positive} \implies \text{new scale limiting access to UV} \]

High momentum (rapidity) particles produce divergences
Cure by introducing a cut-off \( \Lambda = 2W_0(\pi r/\alpha) \) (Lambert function)

\[ \alpha^* < 0: \text{effective size is negative} \implies \text{UV can be probed without issue} \]

Actually “there is more space”
Intuitively explains the hyper-convergence (and Hagedorn)
Consider the $T\bar{T}$ case for $\alpha < 0$

$$\log \left( \frac{\langle \mathcal{O}(0)\mathcal{O}(r) \rangle}{\langle \mathcal{O} \rangle^2} \right) \approx \int_{-\infty}^{\infty} d\theta K_0(2mr \cosh \frac{\theta}{2}) \left| F_2^\theta(\theta; \alpha) \right|^2 + \cdots$$

Expand the Bessel function for $mr \ll 1$

$$\log \left( \frac{\langle \mathcal{O}(0)\mathcal{O}(r) \rangle}{\langle \mathcal{O} \rangle^2} \right) \approx -\log(mr) \int_{-\infty}^{\infty} d\theta f(\theta; \alpha) e^{\frac{\theta}{2} \alpha \sinh \theta} + \cdots = -4\Delta^\theta(\alpha) \log(mr) + \cdots$$
Consider the $\mathbb{T}\mathbb{T}$ case for $\alpha < 0$

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The 2-point functions appear to exhibit power-law scaling at small scales!

Tension with the expectations: there should be no conventional CFT in the UV
Consider the $T\bar{T}$ case for $\alpha < 0$

Consistency check: Zamolodchikov’s $c$-theorem

$$c_{\text{UV}} - c_{\text{IR}} = \frac{3}{2} \int_0^{\infty} dr \, r^3 \langle \Theta(0) \Theta(r) \rangle_{c,\alpha}$$

Zamolodchikov ‘86
Consider the $\bar{T}\bar{T}$ case for $\alpha < 0$

Consistency check: Zamolodchikov’s $c$-theorem

\[
c^{\text{UV}} - c^{\text{IR}} = \frac{3}{2} \int_0^\infty dr \, r^3 \langle \Theta(0) \Theta(r) \rangle_{c,\alpha}
\]

Insert our results in the Ising model case:

\[
c(\alpha) = \frac{3}{8} \int_{-\infty}^{+\infty} dx \frac{\sin^2 \left( \frac{\alpha}{2} \sinh x \right)}{\alpha^2 \cosh^6 \frac{x}{2}} e^{\frac{\alpha}{\pi} x \sinh x}
\]
Consider the $T\bar{T}$ case for $\alpha < 0$

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TBA tells us that $c^{\text{UV}}(\alpha)$ should vanish for all $\alpha < 0$! Can we even define this quantity?
Throwing away all $\beta_t \cosh(t\theta)$ in $\varphi(\theta)$ is a strong assumption!

We have an example where they play a fundamental role: sinh-Gordon
Throwing away all $\beta_t \cosh(t\theta)$ in $\varphi(\theta)$ is a strong assumption!

We have an example where they play a fundamental role: sinh-Gordon

$$S_{shG}(\theta) = \frac{\sinh \theta - i \cos \left(\frac{\pi b}{2}\right)}{\sinh \theta + i \cos \left(\frac{\pi b}{2}\right)} = - \exp \left[ -4i \sum_{k=0}^{\infty} (-1)^{k+1} \frac{\cos \left(\frac{2k+1}{2}\pi b\right)}{2k+1} \sinh((2k+1)\theta) \right]$$

Fine-tuned superposition of solvable irrelevant deformations of thermal Ising

LeClair [2107.02230] | Ahn, LeClair [2205.10905]
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WHAT ABOUT THE COSHES?

Can we write the minimal FF in the form we found above?
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Notice the following (new result?):

$$\log F_{\text{min,shG}}(\theta) = -4 \int_0^{\infty} \frac{dx}{x} \frac{\sinh \left( \frac{x+1}{4} \right) \sinh \left( \frac{x-1}{4} \right) \sinh \frac{x}{2} \cos \left( \frac{x \theta}{\pi} \right)}{\sinh^2 x}$$

$$= \log \left( -i \sinh \frac{\theta}{2} \right) - i \pi \theta - \log (-S_{\text{shG}}(\theta)) + C_{\text{shG}}(\theta)$$

$$C_{\text{shG}}(\theta) = \log \frac{2}{2} - \frac{1 + b}{2} \log \left[ \sin \frac{\pi b}{2} - \cosh \theta \right] - \frac{1 - b}{2} \log \left[ -\sin \frac{\pi b}{2} - \cosh \theta \right] +$$

$$- \frac{i}{4\pi} \left[ \left( \text{Li}_2 \left( i e^{-\theta - i \frac{\pi b}{2}} \right) + \text{Li}_2 \left( -i e^{-\theta - i \frac{\pi b}{2}} \right) \right) + (b \rightarrow -b) \right] + (\theta \rightarrow -\theta)$$
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\[
= \log \left( -i \sinh \frac{\theta}{2} \right) - i \frac{\pi - \theta}{2\pi} \log \left( -S_{\text{shG}}(\theta) \right) + C_{\text{shG}}(\theta)
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C_{\text{shG}}(\theta) = \frac{\log 2}{2} - \frac{1 + b}{2} \log \left[ \sin \frac{\pi b}{2} - \cosh \theta \right] - \frac{1 - b}{2} \log \left[ -\sin \frac{\pi b}{2} - \cosh \theta \right] +
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C_{\text{shG}}(\theta) = \frac{\log 2}{2} - \frac{1 + b}{2} \log \left[ \sin \frac{\pi b}{2} - \cosh \theta \right] - \frac{1 - b}{2} \log \left[ -\sin \frac{\pi b}{2} - \cosh \theta \right] + \\
- \frac{i}{4\pi} \left[ \text{Li}_2 \left( ie^{-\theta - i \frac{\pi b}{2}} \right) + \text{Li}_2 \left( -ie^{-\theta + i \frac{\pi b}{2}} \right) + (b \to -b) \right] + (\theta \to -\theta)
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\[
\log F_{\text{min,shG}}(\theta) = -4 \int_0^\infty \frac{dx}{x} \sinh \left( \frac{1+b}{4} x \right) \sinh \left( \frac{1-b}{4} x \right) \sinh \frac{x}{2} \cos \left( x - \frac{\theta}{\pi} \right) = \\
= \log \left( -i \sinh \frac{\theta}{2} \right) - i \frac{\pi - \theta}{2\pi} \log \left( -S_{\text{shG}}(\theta) \right) + C_{\text{shG}}(\theta)
\]

\[
C_{\text{shG}}(\theta) = \frac{\log 2}{2} - \frac{1 + b}{2} \log \left[ \sin \frac{\pi b}{2} - \cosh \theta \right] - \frac{1 - b}{2} \log \left[ -\sin \frac{\pi b}{2} - \cosh \theta \right] + \\
- i \frac{\pi - \theta}{4\pi} \left[ \text{Li}_2 \left( i e^{-\theta - i \frac{\pi b}{2}} \right) + \text{Li}_2 \left( -i e^{-\theta - i \frac{\pi b}{2}} \right) + (b \to -b) \right] + (\theta \to -\theta)
\]

Ising minimal FF

\[
\frac{i\pi - \theta}{2\pi} \sum_s \alpha_s \sinh(s\theta) + \sum_t \beta_t \cosh(t\theta)
\]
In this case the $\beta_t$ are related to the $\alpha_s$

$$S(\theta) = e^{-i \int_0^\infty \frac{dt}{\tau} g(t) \sin \left( \frac{\theta}{\pi} t \right)} \quad \text{and} \quad g(t) = -t^2 \sum_{n=1}^{\infty} \frac{g_n}{t^2 + n^2\pi^2} \quad \Rightarrow \quad \alpha_s = -\frac{1}{2\pi} \frac{g_s}{s^2}$$

$$F_{\text{min}}(\theta) = e^{i \int_0^\infty \frac{dt}{\tau} g(t) \cos \left( \frac{i\pi - \theta}{\pi} t \right)} \quad \Rightarrow \quad \beta_t = \frac{1}{2\pi} \frac{\alpha_t}{t} - \frac{2}{\pi} t \sum_{s=1, s \neq t}^{\infty} \frac{g_s}{s^2 - t^2}$$
Can we take this as a definition?

\[ S_\alpha(\theta) = \exp \left[ -i \sum_{s \in S} \alpha_s \sinh(s\theta) \right] \implies F_{\text{min}}(\theta) = \exp \left[ -\frac{i\pi - \theta}{2\pi} \sum_s \alpha_s \sinh(s\theta) + \sum_t \beta_t \cosh(t\theta) \right] \]

With

\[ \beta_t = \frac{1}{2\pi} \frac{\alpha_t}{t} - \frac{2}{\pi} \sum_{s=1, s \neq t}^{\infty} \frac{g_s}{s^2 - t^2} \]

For \( T\bar{T} \)-deformed Ising

\[ F_{\text{min}}(\theta) = -\frac{\alpha}{2\pi} \left[ \cosh \theta \log \left( 2 \cosh \theta - 2 \right) + 1 - \theta \sinh \theta + i\pi e^\theta \right] \]
Very general expression for FF in IQFTs deformed by arbitrary “generalised TTbar”

Indeterminacy poses problems: we need to find physical reasons to fix it
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Our representation works for standard IQFTs: study the role of $\beta_s$ there

Attempt comparison with existing results on correlation functions
SOLVING THE FORM FACTOR BOOTSTRAP FOR SOLVABLE IRRELEVANT DEFORMATIONS

CONCLUSIONS AND OUTLOOK

Very general expression for FF in IQFTs deformed by arbitrary “generalised TTbar”

Indeterminacy poses problems: we need to find physical reasons to fix it

Our representation works for standard IQFTs: study the role of $\beta_s$ there

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Extension to “twist fields” and computation of entanglement measures: straightforward

Castro-Alvaredo, Negro, Sailis [2306.11064] | Hou, He, Jiang [2306.07784]

Extension to theories w/ bound states and/or non-diagonal scattering
Thank you

Happy Birthday Fedor!