$\mathrm{Diff}(S^1)$ Coadjoint Orbits,

Universal Teichmüller Space

and

Quantization of Conformal Welding

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October 12, 2023, IHP Paris, *Fedya's 65th birthday conference*

WZW with boundary BSS, 2004

$$Z(h(x)) = \int_{g_{|\partial D}=h} [Dg] e^{ikW(g(z,\bar{z}))} \sim \; e^{i\alpha_2(h_{+|\partial D}, h_{-|\partial D})}$$

$g(z,\bar{z}) : \mathbf{D} \to \mathbf{G}$, $\mathbf{G}$ - some compact Lie group, $\mathbf{D}$ - unit disk.

$h(x) : \partial\mathbf{D} = \mathbf{S^1} \to \mathbf{G}, \quad h(x) = h_+(x) h_-(x)$

h_+, h_- solutions of matrix RH problem: h_+ holomorphic inside & h_- holomorphic outside $\mathbf{D}$ (and $\in \mathbf{G^C}$) - solutions of EoM $\bar{\partial} J = 0$

$\alpha_2(h_1, h_2)$ - $\hat{LG}_k$ group 2-cocycle.

Both sides are "ambiguous".

For Lie algebra: $i\alpha_2(u_+, u_-) = \int_{S^1} Tr\, u_+ du_-, \quad u_+ = u_-^+$.

These follow from cocycle property of $W(g)$ (PW formula)

$$W(gh) = W(g) + W(h) - \int_D Trg^{-1}\partial g\bar{\partial}hh^{-1}$$

For abelian group, say $U(1)$, $g = e^{iX}$, action is of free scalar:

$$W(x) = \int_D \partial X\bar{\partial}X = W(X_0 + X_h(u)) =$$

$$= \int_D \partial X_0\bar{\partial}X_0 + \int_D \partial X_h\bar{\partial}X_h + \int_D (\partial X_0\bar{\partial}X_h + c.c.)$$

$X_{0_{|\partial D}} = 0$, $X_h(u)$ is harmonic $X_{h\partial D} = u$. Cross term vanishes and

$$\int_D \partial X_h\bar{\partial}X_h = i\int_{S^1} u_+ du_-, \quad u = u_+ + u_-, \quad \bar{u}_+ = u_-$$

$WZW : g_{|\partial D} = 1, h = h_+h_-$, cross term $\Rightarrow 0$ in quantum theory.

GravWZ with boundary AS, 2018

Polyakov's (1987) gravitational version (as geometric action $W_{b_0}^{Vir}$ on Virasoro coadjoint orbit Alekseev, SS (1988): $b_0 = -\frac{cn^2}{48\pi^2}$, $n = 0, 1, 2, ...$ are $PSL(2, \mathbf{R}) \cong \text{Möb}(S^1)$ orbits of $\text{Diff}(S^1)$)

$$Z(F(x)) = \int_{f_{|\partial D} = F(x)} [Df] e^{iW_0^{Vir}(f(z,\bar{z}))} \sim \; e^{i\alpha_2(F_{2|\partial D}, F_{1|\partial D})}$$

$$W_0^{Vir}(f) = \int d^2 z \frac{f_{\bar{z}}}{f_z} \left(\frac{f_{zzz}}{f_z} - 2(\frac{f_{zz}}{f_z})^2 \right) = W_{grav}(f^{-1})$$

$F = (F_2 \circ F_1)_{|\partial D}$ - F_2, F_1 defined by F (by EoM $\bar{\partial} S(f) = 0$).

$\alpha_2(F_1, F_2)$ - Bott-Virasoro group 2-cocycle.

These also follow from cocycle property, and bring us to welding.

$\mathrm{Diff}(S^1)/\mathrm{M\ddot{o}b}(S^1)$ and Conformal Welding

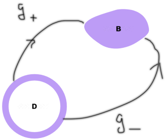
$F \in \mathrm{Diff}(S^1)$ - there exist $g_+, g_- \in \mathrm{Diff}(\mathbf{C})$ s.t. g_+ is analytic inside the disk $\mathbf{D}$ and g_- is analytic outside; on boundary $x \in S^1$

$$g_+(e^{iF(x)}) = g_-(e^{ix})$$

g_+ - maps interior of disk $\mathbf{D}$ to some bounded domain $\mathbf{B} \in \mathbf{C}$

g_- - exterior of the disk $\mathbf{D}^*$ to the compliment of $\mathbf{B}$

Both map S^1 on to boundary $\partial\mathbf{B}$.



g_+, g_- are determined uniquely by F (after fixing three points).

If $F \in \mathrm{Diff}(S^1)$ is replaced by quasiconformal homeomorphism of S^1 one gets $T(1)$ - Universal Teichmüller Space Bers (1965).

g_+, g_- - now are quasiconformal homeomorphisms of $\mathbf{C}$.

$T(1)$ contains all Teichmüller spaces $T(\Gamma)$ of every Fuchsian Γ.

$\mathrm{Diff}(S^1)/\mathrm{M\ddot{o}b}(S^1)$ viewed as a coadjoint orbit of the Bott-Virasoro group is a subspace of $T_0(1)$ - connected component $T(1)$.

$T(1)$ admits a new structure of a complex Hilbert manifold (Takhtajan, Teo (2004)).

$T_0(1)$ - Kahler-Einstein, has analog of WP metric. Kahler potential

$$K_{TT}(g_-, g_+) = K_{TT}(g_+) + K_{TT}(g_-) = \iint_D |\frac{g_+^{''}}{g_+^{'}}|^2 d^2 z + \iint_{D^*} |\frac{g_-^{''}}{g_-^{'}}|^2 d^2 z$$

This helps define real $\alpha_2(g_-, g_+)$ in terms of Bott-Virasoro cocycle:

$$\alpha_2(g_-, g_+) = \alpha_2^{Bott}(g_-, g_+) + iK_{TT}(g_-, g_+), \quad Im\alpha_2 = 0$$

Canonical Transformations, Berezin's Quantization, Welding and Cocycles (AST, 2022)

TT can be directly connected to Berezin (and Segal-Wilson).

Consider bosonic field on a circle $\phi(x)$ and $\Omega = \int_{S^1} \delta\phi \partial_x \delta\phi \, dx$.

Under diffeomorfism $\tilde{x} = F(x)$: $\tilde{\phi}(\tilde{x}) = \phi(x)$ and Ω is invariant.

$$\phi(x) = \sum_n \frac{1}{\sqrt{|n|}} a_n e^{inx} = \sum_n \frac{1}{\sqrt{|n|}} \tilde{a}_n e^{in\tilde{x}}, \quad a_n^+ = a_{-n}$$

$$\Omega = \sum_{n>0} \delta a_n \delta a_n^+ = \sum_{n>0} \delta \tilde{a}_n \delta \tilde{a}_n^+$$

$$[a_n, a_m^+] = [\tilde{a}_n, \tilde{a}_m^+] = \delta_{n,m}$$

This defines the canonical transformation $(a, a^+) \to (\tilde{a}, \tilde{a}^+)$

$$\begin{pmatrix} \tilde{a} \\ \tilde{a}^+ \end{pmatrix} = C \begin{pmatrix} a \\ a^+ \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a \\ a^+ \end{pmatrix}$$

$\alpha, \beta, \gamma, \delta$ are essentially Grunsky matrices of TT and are expressed in terms of $F(x)$. $\alpha^{-1}\beta$ and $\gamma\alpha^{-1}$ are symmetric, $\alpha\delta^t - \beta\gamma^t = 1$.

Fock space **A** with vacuum $|0>, a_n|0>= 0$, Fock space **B** with $|\tilde{0}>, \tilde{a}_n|\tilde{0}>= 0$, $a_n^+, \tilde{a}_n^+$ - creation operators. Normal form of

$$\tilde{a} = V_F a V_F^{-1} \quad \tilde{a}^+ = V_F a^+ V_F^{-1}, \quad |\tilde{0}>= V(a, a^+)|0>$$

is (Berezin (1962)):

$$V_F(a, a^+) = c \exp\{a^+(\alpha^{-1} - 1)a - \frac{1}{2}a^+(\alpha^{-1}\beta)a^+ + \frac{1}{2}a(\gamma\alpha^{-1})a\}$$

Requirement of unitarity: $\gamma = \beta^*, \delta = \alpha^*, c = 1/\det(\alpha\alpha^+)^{\frac{1}{4}}$ and

$$U_F(a, a^+) = \frac{1}{\det(\alpha\alpha^+)^{\frac{1}{4}}} V_F(a, a^+)$$

U_F is well-defined if matrix elements of C satisfy Fredholm (α) and Hilbert-Schmidt (β) conditions - satisfied for F from $T_0(1)$.

$$1. \quad \det(\alpha\alpha^+)^{\frac{1}{4}} = e^{K_{TT}(g_-,g_+)}$$

$$2. \quad U(g_+) = e^{K_{TT}(g_+)}V(g_+), \quad U(g_-^{-1}) = e^{K_{TT}(g_-)}V(g_-^{-1})$$

$$3. \quad V(F) = V(g_-^{-1})V(g_+); \quad U(F) = U(g_-^{-1})U(g_+)$$

$$4. \qquad U(F_1)U(F_2) = e^{i\tilde{\alpha}_2(F_1,F_2)}U(F_1 \circ F_2)$$

$$i\tilde{\alpha}_2(F_1,F_2) = i\alpha_2^{Ber}(F_1,F_2) + K_{TT}(F_1 \circ F_2) - K_{TT}(F_1) - K_{TT}(F_2)$$

$$i\alpha_2^{Ber}(F_1,F_2) = -\log\det(1 + \alpha_2^{-1}\beta_2\gamma_1\alpha_1^{-1})$$

Interestingly $\alpha_2^{Ber}(F_1, F_2)$ is expressed in terms of $g_{1,-}, g_{2,+}$ only.

$\alpha_2(F_1, F_2)$ is, of course, cohomlogical to Bott-Virasoro coycle:

$$\alpha_2^{Bott}(F_1, F_2) = \int_{S^1} \log(F_1'(F_2(x)))' \log(F_2'(x))dx$$

Bosonic, abelian, $\phi(x)$ with $\Omega = \int_{S^1} \delta\phi\partial_x\delta\phi\, dx$ has non-abelian version $g(x) \in \mathrm{Maps}(S^1) \to \mathbf{G}$

$$\Omega = k\int_{S^1} Tr(\delta g g^{-1})\partial_x(\delta g g^{-1})dx$$

invariant under $\mathrm{Diff}(S^1) : \tilde{x} = F(x), \tilde{g}(\tilde{x}) = g(x)$ (recall WZW).

Analog of free field ϕ' is $J(x) = \partial_x g g^{-1}; J(x) = F'(x)\tilde{J}(F(x))$.
$J(x)$ obey current algebra commutation relations with level $= k$.

Commutation relations for ϕ are replaced by exchange relations for $g(x)$ (Poisson bracket straightforwardly follows from this symplectic form: $\{g(x) \underset{,}{\otimes} g(y)\} = g(x) \otimes g(y) r^{\pm}, x > y$ AS, 1989)

$$g(x) \otimes g(y) = g(y) \otimes g(x) R; \quad x > y$$

Currents $J(x)$, as well as $g(x)$, have free field representation - for every positive root Δ_+ there is a free $\beta_{\Delta_+}, \gamma_{\Delta_+}$ system and for every simple root μ free scalar field ϕ_μ with improved stress-tensor.

Now Berezin can be applied to these free fields in order to find the identificator - operator $V_F(J(x)) = V_F(\beta, \gamma, \phi)$ such that:

$$V_F J(x) V_F^{-1} = F'(x) \tilde{J}(x)(F(x))$$

and repeat all above steps. To be continued ....

HAPPY BIRTHDAY!