Diff($S^1$) Coadjoint Orbits, Universal Teichmüller Space and Quantization of Conformal Welding

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October 12, 2023, IHP Paris, Fedya’s 65th birthday conference
WZW with boundary

\[ Z(h(x)) = \int_{g|_{\partial D}=h} [Dg] e^{ikW(g(z,\bar{z}))} \sim e^{i\alpha_2(h_+|_{\partial D}, h_-|_{\partial D})} \]

\( g(z, \bar{z}) : D \to G, \ G - \) some compact Lie group, \( D - \) unit disk.

\( h(x) : \partial D = S^1 \to G, \quad h(x) = h_+(x)h_-(x) \)

\( h_+, h_- \) solutions of matrix RH problem: \( h_+ \) holomorphic inside \& \( h_- \) holomorphic outside \( D \) (and \( \in G^C \)) - solutions of EoM \( \bar{\partial}J = 0 \)

\( \alpha_2(h_1, h_2) - \hat{LG}_k \) group 2-cocycle.

Both sides are "ambiguous".

For Lie algebra: \( i\alpha_2(u_+, u_-) = \int_{S^1} Tr u_+ du_-, \quad u_+ = u_+^\perp. \)
These follow from cocycle property of $W(g)$ (PW formula)

$$W(gh) = W(g) + W(h) - \int_D Tr g^{-1} \partial g \bar{\partial} h h^{-1}$$

For abelian group, say $U(1)$, $g = e^{iX}$, action is of free scalar:

$$W(x) = \int_D \partial X \bar{\partial} X = W(X_0 + X_h(u)) =$$

$$= \int_D \partial X_0 \bar{\partial} X_0 + \int_D \partial X_h \bar{\partial} X_h + \int_D (\partial X_0 \bar{\partial} X_h + c.c.)$$

$X_{0|\partial D} = 0$, $X_h(u)$ is harmonic $X_{h|\partial D} = u$. Cross term vanishes and

$$\int_D \partial X_h \bar{\partial} X_h = i \int_{S^1} u_+ du_-, \quad u = u_+ + u_-, \quad \bar{u}_+ = u_-$$

$WZW : g|_{\partial D} = 1, h = h_+ h_-$, cross term $\Rightarrow 0$ in quantum theory.
Polyakov's (1987) gravitational version (as geometric action $W_{Vir}^0$) on Virasoro coadjoint orbit Alekseev, SS (1988): $b_0 = -\frac{cn^2}{48\pi^2}$, $n = 0, 1, 2, ...$ are $PSL(2, \mathbb{R}) \cong \text{Möb}(S^1)$ orbits of $\text{Diff}(S^1)$)

$$Z(F(x)) = \int_{f|\partial D = F(x)} [Df] e^{iW_{Vir}^0(f(z,\bar{z}))} \sim e^{i\alpha_2(F_2|\partial D, F_1|\partial D)}$$

$$W_{Vir}^0(f) = \int d^2 z \frac{f\bar{z}}{fz} \left( \frac{fzzz}{fz} - 2 \left( \frac{fzz}{fz} \right)^2 \right) = W_{grav}(f^{-1})$$

$F = (F_2 \circ F_1)|_{\partial D}$ - $F_2, F_1$ defined by $F$ (by EoM $\bar{\partial} S(f) = 0$).

$\alpha_2(F_1, F_2)$ - Bott-Virasoro group 2-cocycle.

These also follow from cocycle property, and bring us to welding.
\textbf{Diff}(S^1)/\text{Möb}(S^1) \text{ and Conformal Welding}

\( F \in \text{Diff}(S^1) \) - there exist \( g_+, g_- \in \text{Diff}(C) \) s.t. \( g_+ \) is analytic inside the disk \( D \) and \( g_- \) is analytic outside; on boundary \( x \in S^1 \)

\[ g_+(e^{iF(x)}) = g_-(e^{ix}) \]

\( g_+ \) - maps interior of disk \( D \) to some bounded domain \( B \in C \)

\( g_- \) - exterior of the disk \( D^* \) to the compliment of \( B \)

Both map \( S^1 \) on to boundary \( \partial B \).

\( g_+, g_- \) are determined uniquely by \( F \) (after fixing three points).

If \( F \in \text{Diff}(S^1) \) is replaced by quasiconformal homeomorphism of \( S^1 \) one gets \( T(1) \) - Universal Teichmüller Space Bers (1965).

\( g_+, g_- \) - now are quasiconformal homeomorphisms of \( C \).
\( T(1) \) contains all Teichmüller spaces \( T(\Gamma) \) of every Fuchsian \( \Gamma \).

\( \text{Diff}(S^1)/\text{Möb}(S^1) \) viewed as a coadjoint orbit of the Bott-Virasoro group is a subspace of \( T_0(1) \) - connected component \( T(1) \).

\( T(1) \) admits a new structure of a complex Hilbert manifold (Takhtajan, Teo (2004)).

\( T_0(1) \) - Kahler-Einstein, has analog of WP metric. Kahler potential

\[
K_{TT}(g_- , g_+) = K_{TT}(g_+) + K_{TT}(g_-) = \iint_D |\frac{g''}{g'_i}|^2 d^2z + \iint_{D^*} |\frac{g''}{g'_i}|^2 d^2z
\]

This helps define real \( \alpha_2(g_- , g_+) \) in terms of Bott-Virasoro cocycle:

\[
\alpha_2(g_- , g_+) = \alpha_2^{\text{Bott}}(g_- , g_+) + iK_{TT}(g_- , g_+), \quad \text{Im} \alpha_2 = 0
\]
TT can be directly connected to Berezin (and Segal-Wilson).

Consider bosonic field on a circle $\phi(x)$ and $\Omega = \int_{S^1} \delta \phi \partial_x \delta \phi \, dx$.

Under diffeomorphism $\tilde{x} = F(x)$: $\tilde{\phi}(\tilde{x}) = \phi(x)$ and $\Omega$ is invariant.

$$\phi(x) = \sum_n \frac{1}{\sqrt{|n|}} a_n e^{inx} = \sum_n \frac{1}{\sqrt{|n|}} \tilde{a}_n e^{in\tilde{x}}, \quad a_n^+ = a_{-n}$$

$$\Omega = \sum_{n>0} \delta a_n \delta a_n^+ = \sum_{n>0} \delta \tilde{a}_n \delta \tilde{a}_n^+$$

$$[a_n, a_m^+] = [\tilde{a}_n, \tilde{a}_m^+] = \delta_{n,m}$$

This defines the canonical transformation $(a, a^+) \rightarrow (\tilde{a}, \tilde{a}^+)$
\[
\begin{pmatrix}
  \tilde{a} \\
  \tilde{a}^+
\end{pmatrix}
= C
\begin{pmatrix}
  a \\
  a^+
\end{pmatrix}
= \begin{pmatrix}
  \alpha & \beta \\
  \gamma & \delta
\end{pmatrix}
\begin{pmatrix}
  a \\
  a^+
\end{pmatrix}
\]

\(\alpha, \beta, \gamma, \delta\) are essentially Grunsky matrices of \(TT\) and are expressed in terms of \(F(x)\). \(\alpha^{-1}\beta\) and \(\gamma\alpha^{-1}\) are symmetric, \(\alpha\delta^t - \beta\gamma^t = 1\).

Fock space \(A\) with vacuum \(|0\rangle, a_n|0\rangle = 0\), Fock space \(B\) with \(|\tilde{0}\rangle, \tilde{a}_n|\tilde{0}\rangle = 0\), \(a^+_n, \tilde{a}^+_n\) - creation operators. Normal form of

\[
\tilde{a} = V_F a V_F^{-1} \quad \tilde{a}^+ = V_F a^+ V_F^{-1}, \quad |\tilde{0}\rangle = V(a, a^+)|0\rangle
\]

is (Berezin (1962)):

\[
V_F(a, a^+) = c \exp \left\{ a^+ (\alpha^{-1} - 1) a - \frac{1}{2} a^+ (\alpha^{-1} \beta) a^+ + \frac{1}{2} a (\gamma \alpha^{-1}) a \right\}
\]

Requirement of unitarity: \(\gamma = \beta^*, \delta = \alpha^*, c = 1/\det(\alpha \alpha^+)^{\frac{1}{4}}\) and

\[
U_F(a, a^+) = \frac{1}{\det(\alpha \alpha^+)^{\frac{1}{4}}} V_F(a, a^+)
\]
$U_F$ is well-defined if matrix elements of $C$ satisfy Fredholm ($\alpha$) and Hilbert-Schmidt ($\beta$) conditions - satisfied for $F$ from $T_0(1)$.

1. $\det(\alpha \alpha^+) \frac{1}{4} = e^{K_{TT}(g^{-},g^{+})}$

2. $U(g^{+}) = e^{K_{TT}(g^{+})}V(g^{+})$, $U(g^{-1}) = e^{K_{TT}(g^{-})}V(g^{-1})$

3. $V(F) = V(g^{-1})V(g^{+})$; $U(F) = U(g^{-1})U(g^{+})$

4. $U(F_1)U(F_2) = e^{i\tilde{\alpha}_2(F_1,F_2)}U(F_1 \circ F_2)$

$i\tilde{\alpha}_2(F_1,F_2) = i\alpha_2^{Ber}(F_1,F_2) + K_{TT}(F_1 \circ F_2) - K_{TT}(F_1) - K_{TT}(F_2)$

$i\alpha_2^{Ber}(F_1,F_2) = -\log \det(1 + \alpha_2^{-1} \beta_2 \gamma_1 \alpha_1^{-1})$
Interestingly $\alpha_{2}^{Ber}(F_1, F_2)$ is expressed in terms of $g_{1,-}, g_{2,+}$ only.

$\alpha_2(F_1, F_2)$ is, of course, cohomological to Bott-Virasoro coycle:

$$\alpha_2^{\text{Bott}}(F_1, F_2) = \int_{S^1} \log(F'_1(F_2(x)))' \log(F'_2(x)) \, dx$$

Bosonic, abelian, $\phi(x)$ with $\Omega = \int_{S^1} \delta \phi \partial_x \delta \phi \, dx$ has non-abelian version $g(x) \in \text{Maps}(S^1) \to G$

$$\Omega = k \int_{S^1} \text{Tr}(\delta gg^{-1}) \partial_x (\delta gg^{-1}) \, dx$$

invariant under $\text{Diff}(S^1) : \tilde{x} = F(x), \tilde{g}(\tilde{x}) = g(x)$ (recall WZW).

Analog of free field $\phi'$ is $J(x) = \partial_x gg^{-1}; J(x) = F'(x)\tilde{J}(F(x))$. $J(x)$ obey current algebra commutation relations with level $= k$. 
Commutation relations for $\phi$ are replaced by exchange relations for $g(x)$ (Poisson bracket straightforwardly follows from this symplectic form: $\{g(x) \otimes g(y)\} = g(x) \otimes g(y) r^\pm, x > y$ \textit{AS, 1989})

$$g(x) \otimes g(y) = g(y) \otimes g(x) R; \quad x > y$$

Currents $J(x)$, as well as $g(x)$, have free field representation - for every positive root $\Delta_+$ there is a free $\beta_{\Delta_+}, \gamma_{\Delta_+}$ system and for every simple root $\mu$ free scalar field $\phi_{\mu}$ with improved stress-tensor.

Now Berezin can be applied to these free fields in order to find the identifier - operator $V_F(J(x)) = V_F(\beta, \gamma, \phi)$ such that:

$$V_F J(x) V_F^{-1} = F'(x) \tilde{J}(x)(F(x))$$

and repeat all above steps. To be continued ....
HAPPY BIRTHDAY!