CFT 3-point functions from integrability

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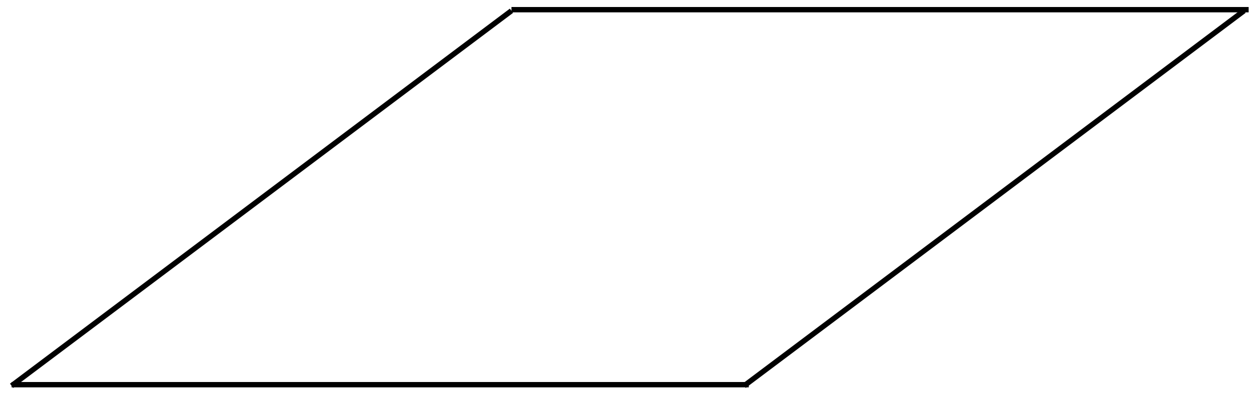
Motivation: **to understand the integrable description of CFT 3pt-functions in terms of Y,T,Q functions in order to generalise it to AdS/CFT**

Characterisation of CFTs

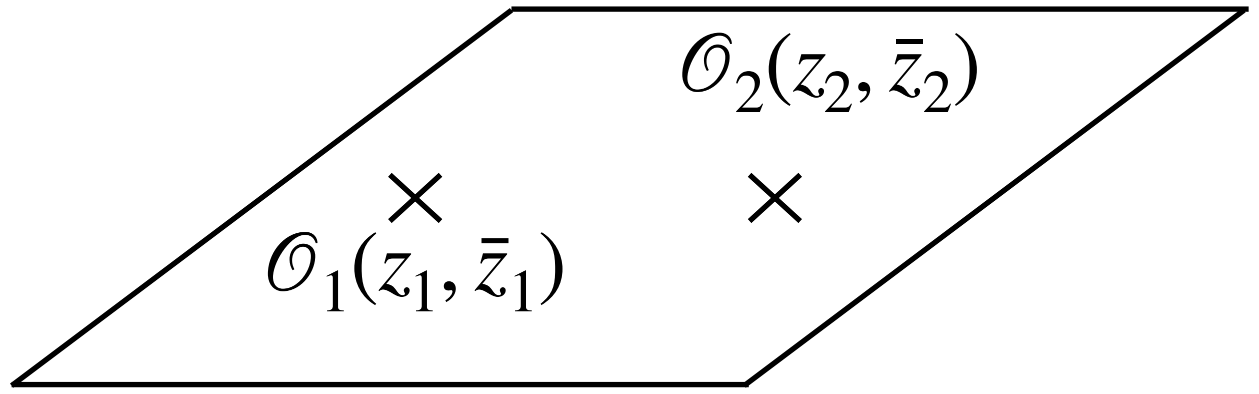


1-point functions

sl2 invariant state

$$\langle 0 | \mathcal{O} | 0 \rangle = 0$$

not always the vacuum (non-unitary CFTs)



2-point functions

conformal dimensions

$$\langle 0 | \mathcal{O}_i(z_i, \bar{z}_i) \mathcal{O}_j(z_j, \bar{z}_j) | 0 \rangle = \delta_{ij} z_{ij}^{-2h} \bar{z}_{ij}^{-2\bar{h}}$$

3-point functions

f: fixed from conformal symmetry

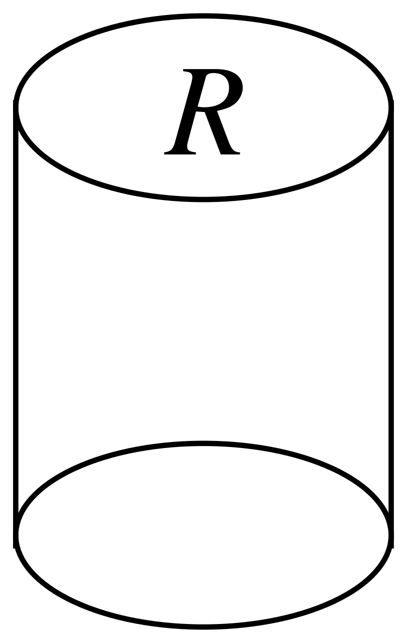
$$\langle 0 | \mathcal{O}_i(z_i, \bar{z}_i) \mathcal{O}_j(z_j, \bar{z}_j) \mathcal{O}_k(z_k, \bar{z}_k) | 0 \rangle = C_{ijk} f(z_{ij}, z_{ik}, z_{jk}) \bar{f}()$$

n-point functions

determined from the OPE coefficients $\mathcal{O}_i(z, \bar{z}) \mathcal{O}_j(0,0) = C_{ijk} \mathcal{O}_k(0,0) z^{h_k - h_i - h_j} \bar{z}^{()}$

AIM: to describe $\boxed{h, \bar{h}, C_{ijk}}$ **from integrability**

CFT on the cylinder



State operator map, $\mathcal{O} \to |\mathcal{O}\rangle$

energy levels

$$E^{\text{CFT}}_{|\mathcal{O}\rangle}(R) = \frac{2\pi}{R}\left[h_{\mathcal{O}} + \bar{h}_{\mathcal{O}} - \frac{c}{24}\right]$$

matrix elements

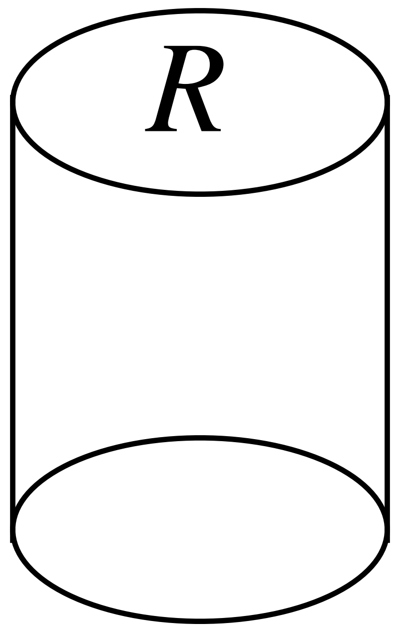
$$\langle \mathcal{O}_1 | \Phi | \mathcal{O}_2 \rangle = \left(\frac{2\pi}{R}\right)^{h+\bar{h}} C_{\mathcal{O}_1 \Phi \mathcal{O}_2}$$

in this talk $\langle \mathcal{O} | \Phi | \mathcal{O} \rangle$

Lee-Yang
Potts
sine-Gordon
reductions

1 component, diagonal
2 component, diagonal
non-diagonal
scattering theories

Idea



Add a massive integrable perturbation and expand at small volume the energy and the matrix element

dimensionful coupling $[\lambda] = 2 - h - \bar{h}$

$$S = S_{\text{CFT}} + \lambda \int_{-\infty}^{\infty} dy \int_0^R dx\, \Phi(x, y)$$

CFT

UV

$\lambda R^{2-h_\Phi-\bar{h}_\Phi}$

RG parameter

IR

massive integrable scattering theory

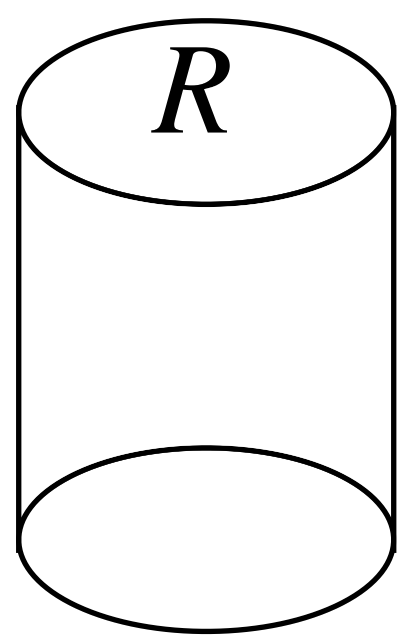
$\mu R^{2-h_\Psi-\bar{h}_\Psi}$

$$S = S_{\text{CFT}} + \mu \int_{-\infty}^{\infty} dy \int_0^R dx\, \Psi(x, y)$$

other massive scattering theory

IR

Implementation



Add a massive integrable perturbation $$S = S_{\rm CFT} + \lambda \int_{-\infty}^{\infty} dy \int_0^R dx\, \Phi(x,y)$$

energy spectrum **expansion in terms of the dimensionful coupling** λ

$$E_{\mathcal{O}}(R,\lambda) = \frac{2\pi}{R}\left[h_{\mathcal{O}} + \bar{h}_{\mathcal{O}} - \frac{c}{24} + \sum_{n=1}^{\infty} d_n \lambda^n \left(\frac{R}{2\pi}\right)^{n(2-2h)}\right]$$

$h < \frac{1}{2}$	$\frac{1}{2} < h < \frac{2}{3}$	$\cdots$
convergent	**1 renormalisation**	
$\epsilon_B R$	$(\epsilon_B + a)R$	

leading term: scaling dimension $h_{\mathcal{O}}$

leading perturbative: λ $$d_1 = 2\pi C_{\mathcal{O}\Phi\mathcal{O}}$$

in unitary theories for the groundstate: λ^2

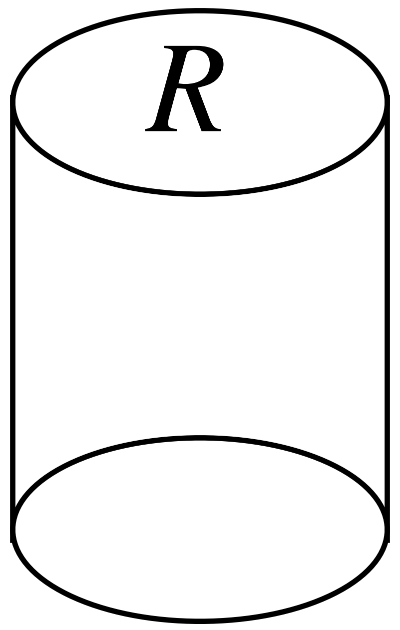
$$d_2(h) = \int_{|z|<1} d^2z (z\bar{z})^{h-1} \langle 0 | \Phi(1,1)\Phi(z,\bar{z}) | 0 \rangle = \frac{\pi}{2} \frac{\Gamma(h)^2 \Gamma(1-2h)}{\Gamma(1-h)^2 \Gamma(2h)}$$

matrix elements

small λ **expansion is a small volume expansion**

$$\langle \mathcal{O} | \Psi | \mathcal{O} \rangle = \left(\frac{2\pi}{R}\right)^{h_\Psi + \bar{h}_\Psi} \left(C_{\mathcal{O}\Psi\mathcal{O}} + \left(\frac{R}{2\pi}\right)^{2-2h} (\dots) \right)$$

Integrable scatterings and the spectrum



Integrable model with one massive particle (Lee-Yang, sinh-Gordon, Bullough-Dodd) [Cardy,Mussardo]

factorized scattering

$$S(\theta) = \frac{\sinh\theta + i\sin A}{\sinh\theta - i\sin A} \cdot \ldots \qquad p = m\sinh\theta$$

groundstate energy as the function of $r = mR$

$$\mathcal{E}_0(R,m) = -m\int \frac{d\theta}{2\pi}\cosh\theta\ \log(1+e^{-\epsilon(\theta)})$$

[Al. Zamolodchikov]

Thermodynamic Bethe Ansatz

$$\epsilon(\theta) = r\cosh\theta - \int \frac{d\theta'}{2\pi}\varphi(\theta-\theta')\log(1+e^{-\epsilon(\theta')}) \qquad \varphi(\theta) = -i\partial_\theta \log S(\theta)$$

generalisations for excited states with extra source terms

different variables and conventions in UV and IR

$$E_0(R,\lambda) - \epsilon_B R = \mathcal{E}_0(R,m) \qquad \lambda = \kappa m^{2-2h}$$

mass-gap relation

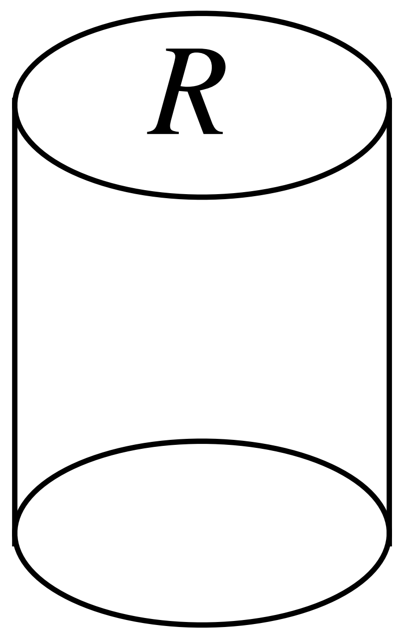
small volume expansion of the TBA energy

$$R\mathcal{E}_{\mathcal{O}}(r) = \epsilon_0 - \frac{\epsilon_B}{m^2}r^2 + \epsilon_1 r^\alpha + \ldots$$

massgap $\quad \kappa^2 = \frac{(2\pi)^{2(2h-2)}}{2\pi d_2(h)} \cdot \epsilon_1$

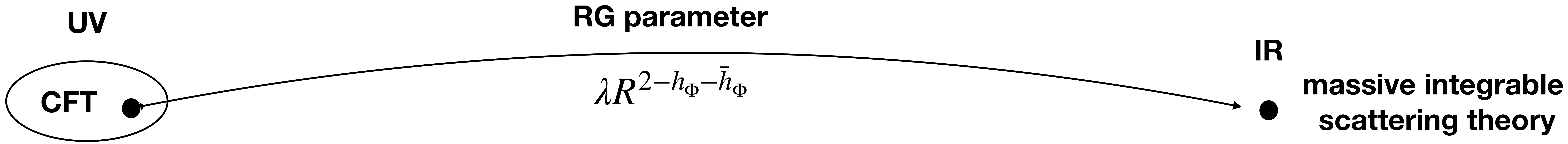
3-pt function $\quad C_{\mathcal{O}\Phi\mathcal{O}} = \frac{1}{2\pi(2\pi\kappa)^{2h-2}} \cdot \epsilon_1$

Summary of the idea



Add a massive integrable perturbation and expand at small volume the energy and the matrix element

$$S = S_{\rm CFT} + \lambda \int_{-\infty}^{\infty} dy \int_0^R dx\, \Phi(x,y)$$



conformal perturbation theory

$$\epsilon(\theta) = r\cosh\theta - \int \frac{d\theta'}{2\pi}\varphi(\theta-\theta')\log(1+e^{-\epsilon(\theta')})$$

$$RE_{\mathcal{O}}(R,\lambda) = 2\pi\left(h_{\mathcal{O}} + \bar{h}_{\mathcal{O}} - \frac{c}{24} + 2\pi C_{\mathcal{O}\Phi\mathcal{O}}\lambda\left(\frac{R}{2\pi}\right)^{(2-2h)} + d_2\lambda^2\left(\frac{R}{2\pi}\right)^{2(2-2h)} + \ldots\right)$$

$$R\mathcal{E}_{\mathcal{O}}(r) = \epsilon_0 - \frac{\epsilon_B}{m^2}r^2 + \epsilon_1 r^\alpha + \ldots$$

mass-gap $\lambda = \kappa m^{2-2h}$ $\qquad \kappa^2 = \dfrac{(2\pi)^{2(2h-2)}}{2\pi d_2(h)} \cdot \epsilon_1$

3-pt function $C_{\mathcal{O}\Phi\mathcal{O}} = \dfrac{1}{2\pi(2\pi\kappa)^{2h-2}} \cdot \epsilon_1$

Integrable scatterings and expectation values

Integrable model with one massive particle

$$\langle 0|\Psi|0\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_i \int \frac{d\mu(\theta_i)}{2\pi} F_c^{\Psi}(\theta_1, \ldots, \theta_n)$$

LeClair-Mussardo formula

$$d\mu(\theta) = \frac{d\theta}{1 + e^{\epsilon(\theta)}}$$

form factor depends on the rapidity differences!

resummations a'la Smirnov [Negro, Smirnov] [Bajnok,Smirnov]

$$\mathcal{G}_n(\theta) = e^{n\theta} + \int \frac{d\mu(\theta')}{2\pi} \varphi(\theta - \theta') \mathcal{G}_n(\theta')$$

$$\mathcal{G}_n = e^{n\theta} + \varphi \circ \mathcal{G}_n$$

dressed by volume corrections

$$\mathcal{G}_n(\theta) = e^{n\theta} + \varphi(\theta - \theta') \circ e^{n\theta'} + \ldots = \frac{1}{1 - \varphi(\theta - \theta') \circ} e^{n\theta'} =: (e^{n\theta})^{\mathrm{dr}}$$

Smirnov: general operators are built from

$$\omega_{n,m} = e^{n\theta} \circ (e^{m\theta})^{\mathrm{dr}}$$

conserved charges and currents of spin s $\omega_{s,1} \quad \omega_{s,-1}$

general vertex operators

$$\frac{\langle 0|e^{(a+b)\phi}|0\rangle}{\langle 0|e^{a\phi}|0\rangle} = \omega^a_{1,-1} + \mathrm{const}$$

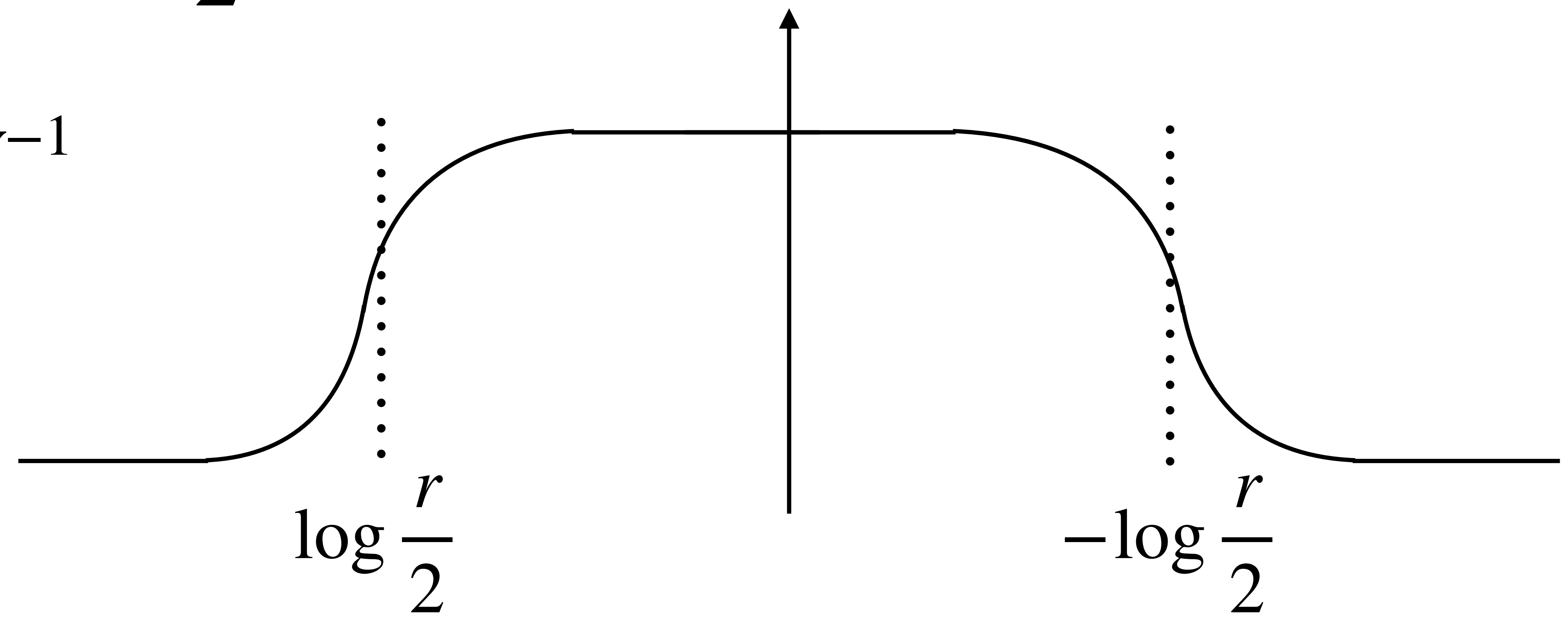
$$\mathcal{G}^a_n = e^{n\theta} + \varphi_a \circ \mathcal{G}^a_n$$

deformation of the kernel

Small volume solution of the TBA

$$\log Y = \frac{r}{2}e^{\theta} + \frac{r}{2}e^{-\theta} - \varphi \star \log\left(1 + Y^{-1}\right) \qquad Y(\theta) = e^{\epsilon(\theta)}$$

schematic solution for Y^{-1}



anti-kink solution

$$\log Y_A = \frac{r}{2}e^{-\theta} - \varphi \star \log\left(1 + Y_A^{-1}\right)$$

kink solution

$$\log Y_K = \frac{r}{2}e^{\theta} - \varphi \star \log\left(1 + Y_K^{-1}\right)$$

conformal anti-kink solution

$$Y_A(\theta) \equiv Y_-\left(\theta - \log\frac{r}{2}\right)$$

$$\log Y_- = e^{-\theta} - \varphi \star \log\left(1 + Y_-^{-1}\right)$$

conformal kink solution

$$Y_K(\theta) \equiv Y_+\left(\theta + \log\frac{r}{2}\right)$$

$$\log Y_+ = e^{\theta} - \varphi \star \log\left(1 + Y_+^{-1}\right)$$

$$Y_\pm(\theta + i\pi/3)Y_\pm(\theta - i\pi/3) = 1 + Y_\pm(\theta)$$

Spectrum from integrability (Lee-Yang)

integrable description of CFTs: [Bazhanov, Lukyanov, Zamolodchikov]

from TBA

from lattice

[Bajnok, el Deeb, Pearce]

$$Y_\pm(s + i\pi/3)Y_\pm(s - i\pi/3) = 1 + Y_\pm(s)$$

asymptotics

$$Y_+(s) \sim_{s\to+\infty} \exp(e^s) \qquad Y_-(s) \sim_{s\to-\infty} \exp(e^{-s})$$

analytical properties for the ground state

$$\epsilon_\pm(s) = e^{\pm s} - \varphi \star \log\left(1 + e^{-\epsilon_\pm}\right) \qquad Y_\pm(s) = e^{\epsilon_\pm(s)}$$

$$h_0 + \bar{h}_0 - \frac{c}{12} = E_+ + E_- \qquad E_\pm = -\int_{-\infty}^{\infty} \frac{ds}{2\pi} e^{\pm s} \log\left(1 + e^{-\epsilon_\pm}\right)$$

dilogarithm trick

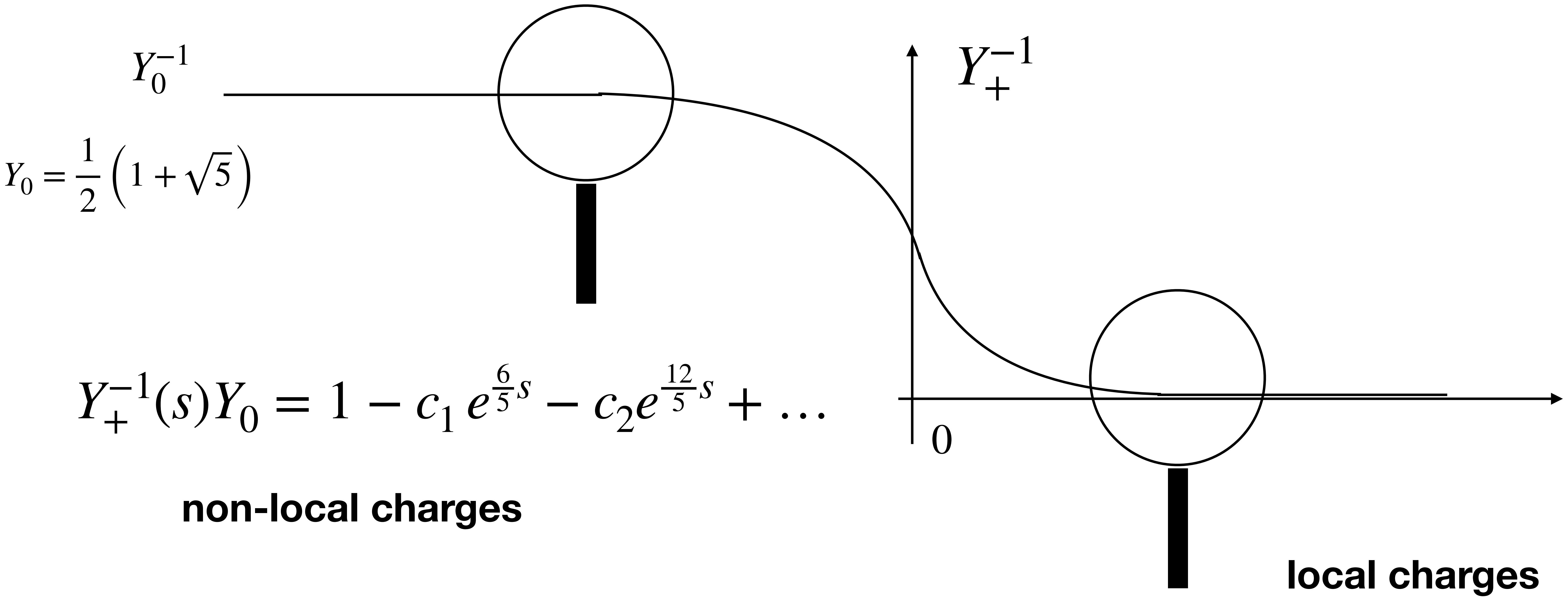
$$E_+ + E_- = -\frac{1}{30} \qquad h = -\frac{1}{5} \quad c = -\frac{22}{5}$$

Can be extended for excited states, boundaries, defects [Bajnok, el Deeb, Pearce]

Aim

Express the 3-point functions in terms of these integrable data

In particular $C_{\mathcal{O}\Phi\mathcal{O}}$ **in terms of** $Y_{\pm}(s) = T_{\pm}(s)$



$$Y_+^{-1}(s)Y_0 = 1 - c_1\, e^{\frac{6}{5}s} - c_2 e^{\frac{12}{5}s} + \ldots$$

non-local charges

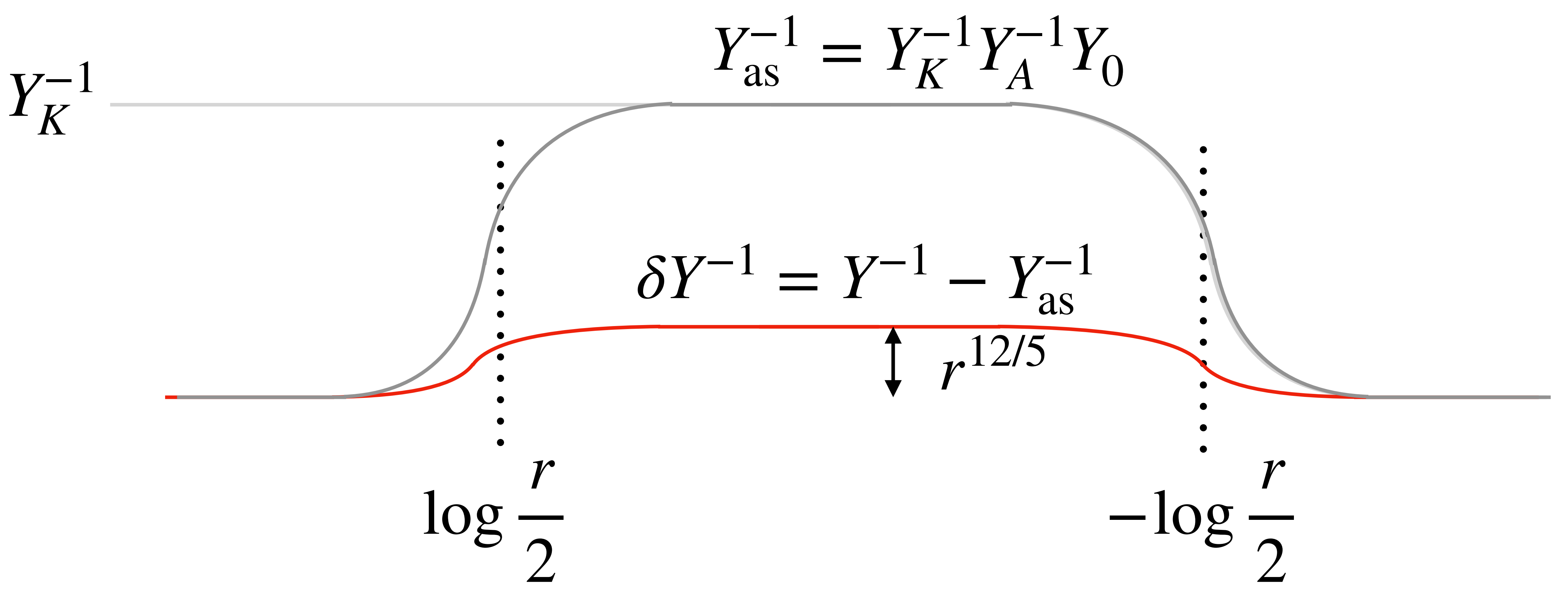
[Bazhanov, Lukyanov, Zamolodchikov]

local charges

$$Y_+^{-1}(s) = \tilde{c}_1\, e^{-e^s} + \ldots$$

Small volume expansion of the TBA

exact vs asymptotic solution



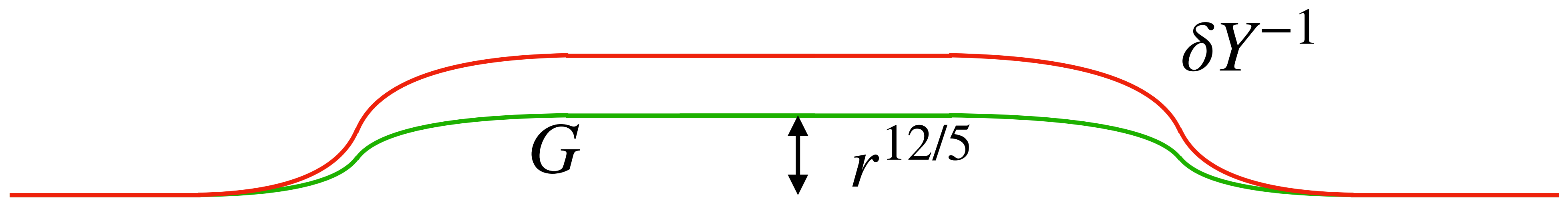
linearised equation for δY

$$-Y_{\rm as}\delta Y^{-1} = \underbrace{\frac{r}{2}e^{\theta} + \frac{r}{2}e^{-\theta} - \log Y_{\rm as} - \varphi \star \log(1 + Y_{\rm as}^{-1})}_{\text{source}} - \varphi \star \frac{\delta Y^{-1}}{1 + Y_{\rm as}^{-1}}$$

$$\text{source} = -\varphi \star \left[\log(1 + Y_{\rm as}^{-1}) - \log(1 + Y_K^{-1}) - \log(1 + Y_A^{-1}) + \log Y_0\right] \equiv -\varphi \star G$$

Source in the linearised equation

exact vs asymptotic solution



$$G = \log(1 + Y_{\mathrm{as}}^{-1}) - \log(1 + Y_K^{-1}) - \log(1 + Y_A^{-1}) + \log Y_0$$

LO volume dependence

$$G = \log \frac{1 + Y_K^{-1} Y_A^{-1} Y_0}{1 + Y_K^{-1}} - \log \frac{1 + Y_0^{-1} Y_A^{-1} Y_0}{1 + Y_0^{-1}}$$

$$Y_A^{-1} Y_0 \sim 1 - c_1 e^{-\frac{6}{5}\left(s - 2\log\frac{r}{2}\right)} = 1 - c_1 \left(\frac{r}{2}\right)^{\frac{12}{5}} e^{-\frac{6}{5}s}$$

$$G_+(s) = \lim_{r\to 0} r^{-\frac{12}{5}} G\left(s - \log\frac{r}{2}\right)$$

kink correction

$$Y_+\, \delta Y_+^{-1} = \varphi \star \left[G_+ + \frac{\delta Y_+^{-1}}{1 + Y_+^{-1}}\right]$$

correction source

$$G_+(s) = -\left(\frac{1}{1 + Y_+} - \frac{1}{1 + Y_0}\right) \cdot c_1\, 2^{-\frac{12}{5}} e^{-\frac{6}{5}s}$$

Energy formula

$$\mathscr{E}_0(r)/m = -\int \frac{d\theta}{2\pi}\cosh\theta \log\left(1+Y_{\rm as}^{-1}+\delta Y^{-1}\right) = \frac{2\pi}{r}(E_+ + E_-) - \frac{\epsilon_B}{m^2} r + \epsilon_1 r^{\frac{12}{5}} + \dots$$

$$\log\left(1+Y_{\rm as}^{-1}\right)$$

$$+\frac{\delta Y^{-1}}{1+Y_{\rm as}^{-1}} + G$$

$$\log(1+Y_K^{-1}) + \log(1+Y_A^{-1}) - \log(1+Y_0^{-1})$$

$$-\int \frac{ds}{2\pi} e^s \left[G_+ + \frac{\delta Y_+^{-1}}{1+Y_+^{-1}}\right]$$

central charge | **bulk energy constant** | **subleading energy correction**

$$-\frac{2}{r}\int \frac{ds}{2\pi} e^s \log\left(1+Y_+^{-1}\right)$$

$$\frac{r}{2}\int \frac{ds}{2\pi} e^s \frac{\partial \log Y_-}{1+Y_-}$$

$$r^{\frac{12}{5}}\int \frac{ds}{2\pi} G_+ \cdot \partial \log Y_+$$

total energy correction

$$\epsilon_1 = \int \frac{ds}{2\pi} G_+ \cdot \partial \log Y_+ + \int \frac{ds}{2\pi} G_- \cdot \partial \log Y_-$$

3pt-function

$$C_{\Phi\Phi\Phi} = (\dots) \cdot \left[\int \frac{ds}{2\pi} G_+ \cdot \partial \log Y_+ + \int \frac{ds}{2\pi} G_- \cdot \partial \log Y_-\right]$$

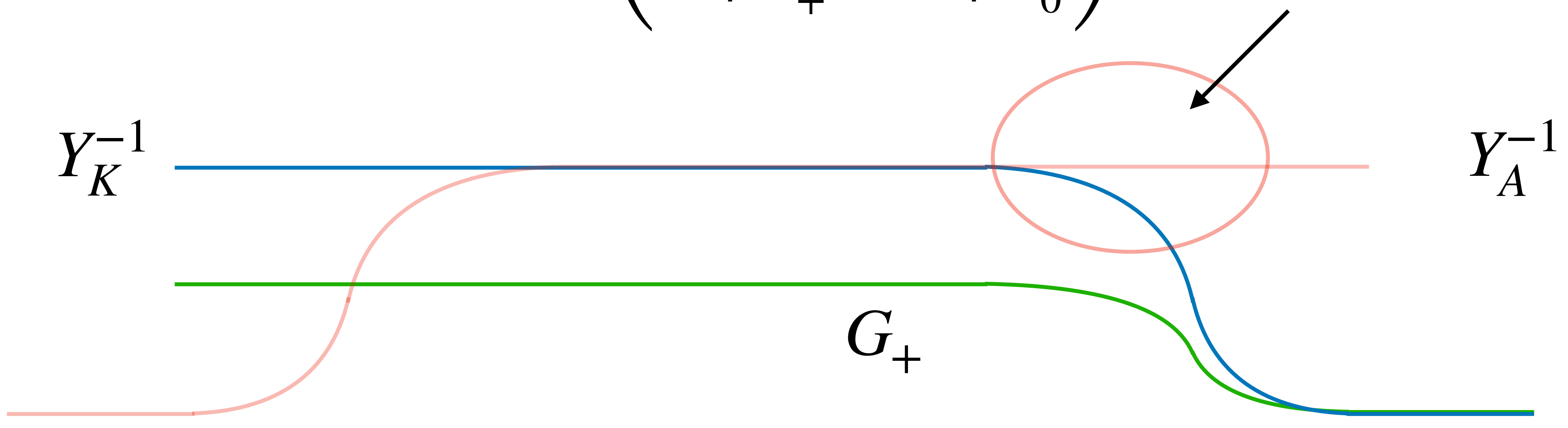
3-pt functions

$$\mathcal{E}_0(r)/m = -\int \frac{d\theta}{2\pi} \cosh\theta \boxed{\log\left(1 + Y_{\text{as}}^{-1} + \delta Y^{-1}\right)} = \frac{2\pi}{r}(E_+ + E_-) - \frac{\epsilon_B}{m^2} r + \epsilon_1 r^{\frac{12}{5}} + \ldots$$

3pt-function $C_{\Phi\Phi\Phi} = (\ldots) \cdot \left[\int \frac{ds}{2\pi} G_+ \cdot \partial \log Y_+ + \int \frac{ds}{2\pi} G_- \cdot \partial \log Y_-\right]$

depends only on CFT quantities
interaction between kink and anti-kink

$$G_+(s) = -\left(\frac{1}{1+Y_+} - \frac{1}{1+Y_0}\right) \cdot c_1\, 2^{-\frac{12}{5}} e^{-\frac{6}{5}s}$$



Other 3-point functions $C_{1\Phi 1} = 0$

expand excited states energies

$$Y_\pm^{-1}(s) Y_\pm = 1 - c_1' e^{\pm\frac{12}{5}s} + \ldots$$

excited states

excited states by analytical continuation or from the lattice [Dorey, Tateo]

$$\log Y = r\cosh\theta + \sum_i \eta_i \log S(\theta-\theta_i) - \varphi \star \log(1+Y^{-1}) \qquad Y(\theta_i) = -1$$

$$\mathcal{E}_1 = -im\sum_i \eta_i \sinh\theta_i - m\int \frac{d\theta}{2\pi}\cosh\theta\log(1+Y^{-1})$$

small volume expansion, kink equations

$$\theta_i^+ = s_i - \log\frac{r}{2}$$

$$\log Y_\pm = e^{\pm\theta} + \sum_i \eta_i \log S(\theta-\theta_i^\pm) - \varphi\star\log(1+Y_\pm^{-1}) \qquad Y_\pm^{-1}(s)Y_\pm = 1 - c_1' e^{\pm\frac{12}{5}s} + \dots$$

$$R\mathcal{E}_1 = 2\pi\frac{11}{30} - \frac{\epsilon_B}{m^2}r^2 + \epsilon_1 r^{\frac{24}{5}} + \dots \qquad \epsilon_1 = -i\tilde{c}2^{-\frac{24}{5}}\sum_i \eta_i e^{-\frac{12}{5}s_i} + \int\frac{ds}{2\pi}G_+\partial\log Y_+ + (+\leftrightarrow-)$$

mass-gap relation for the Lee-Yang model

$$G_+(s) = \left(\frac{1}{1+Y_+(s)} - \frac{1}{1+Y_0}\right)\tilde{c}_1 2^{-\frac{24}{5}}e^{-\frac{12}{5}s}$$

excited states 3pt functions: analytical continuation of the ground-state ones

Potts model perturbed with $\Phi_{1,2}$

2 particles, 2-component TBA

$$\log Y_i = r\cosh\theta - \varphi_{ij} \star \log(1+Y_j^{-1})$$

$$\varphi_{11} = \varphi_{22} = -\frac{\sqrt{3}}{1+2\cosh\theta} \quad ; \qquad \varphi_{12} = \varphi_{21} = -\frac{\sqrt{3}}{-1+2\cosh\theta}$$

$$E_0^{Potts}(R) = -m\sum_{i=1}^{2}\int\frac{d\theta}{2\pi}\cosh\theta\,\log(1+Y_i(\theta)^{-1}) = 2E_0^{LY}(R) = \frac{2\pi}{R}\left(-\frac{1}{15} + 2\pi d_2\lambda^2\left(\frac{R}{2\pi}\right)^{\frac{12}{5}} + \dots\right)$$

massgap

$$\kappa^2 = \frac{(2\pi)^{2(2h-2)}}{2\pi d_2(h)}\cdot\epsilon_1$$

twisted ground-state

$$\log Y = \frac{2i\pi}{3} + r\cosh\theta - \varphi_{11}\star\log(1+Y^{-1}) - \varphi_{12}\star\log(1+\bar{Y}^{-1})$$

$$Y_+(\theta)Y_0^{-1} = 1 + ice^{\frac{3}{5}\theta} + \dots$$

$$G_+(s) = \left\{\frac{1}{1+Y_+(s)} - \frac{1}{1+Y_0}\right\} ic(1/2)^{\frac{6}{5}}e^{-\frac{3}{5}s}$$

$$E_\sigma^{Potts} = \frac{2\pi}{R}\left(\frac{1}{15} + 2\pi C_{\sigma\Phi\sigma}\lambda\left(\frac{R}{2\pi}\right)^{\frac{6}{5}} + \dots\right)$$

3-pt function

$$C_{\sigma\Phi\sigma} = \frac{1}{\kappa(2\pi\kappa)^{2h}}\cdot\epsilon_1$$

structure constant

$$C_{\sigma\Phi\sigma} \propto \int\frac{d\theta}{2\pi}G_+\partial\log Y_+ + \text{cc}\,.$$

general excited state: deformation of the contour

sine-Gordon model and its reductions

Destri de Vega equation (analogue of TBA)

$$Z(\theta) = MR\sinh\theta + \alpha + \int_{-\infty}^{\infty} \frac{dx}{2\pi i}\phi(\theta - x - i\eta)\log(1 + e^{iZ(x+i\eta)}) - cc\,.$$

$$\phi(\theta) = \int dk\, e^{i2k\theta}\frac{\sinh(\pi(p-1)k)}{\sinh(\pi pk)\cosh(\pi k)}$$

energy

$$\mathcal{E}_0 = -M\int \frac{dx}{2\pi i}\sinh(x + i\eta)\log(1 + e^{iZ(x+i\eta)}) - cc\,.$$

UV limit

$$Z_\pm(\theta) = \pm e^{\pm\theta} + \alpha + \dots$$

the tail

$$Z_\pm(\theta) = Z_0 + c_1^\pm e^{\pm\frac{2}{1+p}\theta} + c_2^\pm e^{\pm\frac{4}{1+p}\theta} + \dots$$

$$R\mathcal{E}_0 = \epsilon_0 - \epsilon_B r^2 + \epsilon_1 r^{1+4/(1+p)} + \dots$$

$$\epsilon_1 = -\int \frac{dx}{2\pi i}\left\{G_+(x+i\eta)\partial_x Z_+(x+i\eta) - \frac{1}{1+e^{-iZ_0}}g_-(x+i\eta)\exp(x+i\eta)\right\} - cc\,.$$

$$G_+(x) = \left\{\frac{1}{1+e^{-iZ_+(x)}} - \frac{1}{1+e^{-iZ_0}}\right\}g_-(x) \qquad g_-(x) = ic_1^-\ 2^{-\frac{4}{1+p}}\ e^{-\frac{2}{1+p}x}$$

for specific p, α, e.g. Potts excited state $p = 5;\ \alpha = 3\pi/5$ we have $Z_0 = \pi$ and we had to regularise G_+

sine-Gordon model: other operators

expectation values of local operators using the fermonic basis [Boos,Jimbo,Miwa,Smirnov]

$$\mu = M^{\nu}\left(\frac{\sqrt{\pi}\Gamma(\frac{1}{2\nu})}{2\Gamma(\frac{1-\nu}{2\nu})}\right)^{\nu}\frac{1}{\Gamma(\nu)} \qquad \nu^{-1} = p+1$$

$$\langle : e^{\beta\phi} : \rangle = C_1(0)\mu^{-2+2/\nu}(\omega_{1,-1} + \varphi_1)$$

$$\varphi_0 = -\frac{1}{\nu^2}\cot\frac{\pi}{2\nu}\cot\frac{3\pi}{2\nu} \quad ; \quad \varphi_1 = \frac{i}{\nu}\cot\frac{\pi}{2\nu} \quad ; \quad \varphi_2 = \frac{i}{\nu}\cot\frac{3\pi}{2\nu}$$

$$\langle : e^{2\beta\phi} : \rangle = C_2(0)\mu^{\frac{8}{\nu}-8}\big(\varphi_0 + \varphi_1\omega_{3,-3} + \varphi_2\omega_{1,-1} + \omega_{1,-1}\omega_{3,-3} - \omega_{1,-3}\omega_{3,-1}\big) \qquad \omega_{n,m} = \frac{1}{\nu}e^{n\theta}\circ\mathcal{G}_m(\theta)$$

$$\mathcal{G}_m(\theta) = e^{m\theta} - \int dt\,\varphi(\theta - t - i\epsilon)\frac{e^{iZ(t+i\epsilon)}}{1+e^{iZ(t+i\epsilon)}}\mathcal{G}_m(t+i\epsilon) - \int dt\,\varphi(\theta - t + i\epsilon)\frac{e^{-iZ(t-i\epsilon)}}{1+e^{-iZ(t-i\epsilon)}}\mathcal{G}_m(t-i\epsilon)$$

$$\mathcal{G}_m(\theta) = e^{m\theta} + \varphi(\theta - t)\circ\mathcal{G}_m(t)$$

UV limit $\omega_{2k-1,1-2j} \sim (r)^{2-2k-2j+4\nu}$ **subleading calculation is needed**

Conclusions

small volume expansion of the TBA energy

$$R\mathscr{E}_{\mathscr{O}}(r) = \epsilon_0 - \frac{\epsilon_B}{m^2} r^2 + \epsilon_1 r^\alpha + \dots$$

central charge

$$\epsilon_0 \sim -\int \frac{ds}{2\pi} e^s \log\left(1 + Y_+^{-1}(s)\right)$$

bulk energy constant

$$\epsilon_B \sim \int \frac{ds}{2\pi} e^{-s} \frac{\partial \log Y_+(s)}{1 + Y_+(s)}$$

We managed to calculate ϵ_1 in terms of CFT quantities

$$\epsilon_1 \sim c_1 \int \frac{ds}{2\pi} e^{-(1-h)s} \left(\frac{1}{1 + Y_+(s)} - \frac{1}{1 + Y_0} \right) \cdot \partial \log Y_+(s)$$

$$Y_-^{-1}(s) Y_0 = 1 - c_1\, e^{-(1-h)s} + \dots$$

By comparing to PCFT

ground state massgap

$$\kappa^2 \sim \frac{\Gamma(1-h)^2 \Gamma(2h)}{\Gamma(h)^2 \Gamma(1-2h)} \epsilon_1$$

excited state 3-pt function

$$C_{\mathscr{O}\Phi\mathscr{O}} \sim \frac{1}{\kappa} \cdot \epsilon_1$$

We showed how it works for the ground and excited states in

Lee-Yang
Potts
sine-Gordon
reductions

1 component, diagonal
2 component, diagonal
non-diagonal
scattering theories

Open problems, future research

Explicit evaluation of the integral, similarly to the central charge and bulk energy constant

Extension of the formulas for excited states in the sine-Gordon model and its reductions

Deformation of the kernel al'a Smirnov to describe other operators
(results for small deformations, problem with analytical continuation)

Reformulations in terms of T,Q functions (Tateo)

Non-diagonal form factors in finite volume (with Fedor in the sinh-Gordon)

Generalizations for AdS/CFT

Happy birthday!